Important topics in Chapter 6 (Gödel's Incompleteness Theorem)
$\mathcal{N}$ is the *first order system* of arithmetic in the language $\mathcal{L}_{\mathcal{N}}$.
N is the (intended) *interpretation* of $\mathcal{L}_{\mathcal{N}}$ in which $D_N = \mathbb{N}$.

Notation:
$0^{(n)}$ is an abbreviation for the *closed term* in $\mathcal{L}_{\mathcal{N}}$ that is interpreted in N by the natural number n.
In particular, $0^{(n)}$ abbreviates the closed term $f_1^1(...f_1^1(a_1)...)$ in which f_1^1 occurs n times.
Comment: In N, a_1 is interpreted as 0, and f_1^1 (successor function) is abbreviated by $'$. So $0^{(n)}$ is an abbreviation for the term in $\mathcal{L}_{\mathcal{N}}$ corresponding to the number $0''^{\prime...\prime}$ (n primes) in $\mathbb{N}$; *i.e.*, the nth successor of 0, which is the number n.

Def. 6.3. A k-place **relation** R defined on $\mathbb{N}$ is **expressible in $\mathcal{N}$** if there is a *wf* $\mathcal{A}(x_1, ..., x_k)$ of $\mathcal{L}_{\mathcal{N}}$ with k free variables such that for any $n_1, ..., n_k \in \mathbb{N}$,
(i) If $R(n_1, ..., n_k)$ holds in $\mathbb{N}$, then $\vdash_{\mathcal{N}} \mathcal{A}(0^{(n_1)}...0^{(n_k)})$
(ii) If $R(n_1, ..., n_k)$ doesn't hold in $\mathbb{N}$, then $\vdash_{\mathcal{N}} \sim\mathcal{A}(0^{(n_1)}...0^{(n_k)})$

Def. 6.5. A k-place **function** f defined on $\mathbb{N}$ is **expressible in $\mathcal{N}$** if there is a *wf* $\mathcal{A}(x_1, ..., x_{k+1})$ of $\mathcal{L}_{\mathcal{N}}$ with k+1 free variables such that for any $n_1, ..., n_{k+1} \in \mathbb{N}$,
(i) If $f(n_1, ..., n_k) = n_{k+1}$, then $\vdash_{\mathcal{N}} \mathcal{A}(0^{(n_1)}...0^{(n_{k+1})})$
(ii) If $f(n_1, ..., n_k) = n_{k+1}$, then $\vdash_{\mathcal{N}} \sim\mathcal{A}(0^{(n_1)}...0^{(n_{k+1})})$
(iii) $(\exists_1 x_{k+1})\mathcal{A}(0^{(n_1)}...0^{(n_{k+1})}, x_{k+1})$

Props. 6.10, 6.11. Not all relations and functions defined on $\mathbb{N}$ are expressible in $\mathcal{N}$.

Prop. 6.12. A relation (or function) defined on $\mathbb{N}$ is expressible in $\mathcal{N}$ *iff* it is recursive.

Recursive functions:

I. Basic recursive functions
1. The *zero function.* $z : \mathbb{N} \to \mathbb{N}$, $z(n) = 0$.
2. The *successor function.* $s : \mathbb{N} \to \mathbb{N}$, $s(n) = n + 1$.
3. *Projection functions.* $p_i^k : \mathbb{N}^k \to \mathbb{N}$, $p_i^k(n_1, ..., n_k) = n_i$, for all $n_1, ..., n_k \in \mathbb{N}$, $k \geq 1$, $i = 1...k$.

II. Rules of Formation
1. ***Composition:*** If $g : \mathbb{N}^j \to \mathbb{N}$ and $h_i : \mathbb{N}^k \to \mathbb{N}$, $1 \leq i \leq j$, are recursive functions, so is $f : \mathbb{N}^k \to \mathbb{N}$, given by
$$f(n_1, ..., n_k) = g(\mathrm{h}_1(n_1, ..., n_k), ..., h_j(n_1, ..., n_k))$$
2. ***Recursion:*** If $g : \mathbb{N}^k \to \mathbb{N}$ and $h : \mathbb{N}^{k+2} \to \mathbb{N}$ are recursive functions, so is $f : \mathbb{N}^{k+1} \to \mathbb{N}$, given by
$$f(n_1, ..., n_k, 0) = g(n_1, ..., n_k)$$
$$f(n_1, ..., n_k, n{+}1) = h(n_1, ..., n_k, n, f(n_1, ..., n_k, n))$$
Comment: For $k = 0$, the function $g : \mathbb{N}^0 \to \mathbb{N}$ is simply an element of $\mathbb{N}$.
3. ***Least Number Operator μ:*** If $g : \mathbb{N}^{k+1} \to \mathbb{N}$ is a recursive function and has the property that for all $n_1, ..., n_k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $g(n_1, ..., n_k, n) = 0$, then $f : \mathbb{N}^k \to \mathbb{N}$ is recursive, where
$$f(n_1, ..., n_k) = \min\{n \in \mathbb{N} \mid g(n_1, ..., n_k, n) = 0\}$$

Def. 6.15. A **function on $\mathbb{N}$ is recursive** if it can be obtained from the basic recursive functions by a finite number of applications of the rules of formation.

Def. 6.17. Let R be a k-place relation on $\mathbb{N}$. The **characteristic function** of R, denoted by c_R, is defined by

$$c_R(n_1, \dots, n_k) = \begin{cases} 0 \text{ if } R(n_1, \dots, n_k) \text{ holds} \\ 1 \text{ if } R(n_1, \dots, n_k) \text{ doesn't hold} \end{cases}$$

Def. 6.18. A **relation on $\mathbb{N}$ is recursive** if its characteristic function is recursive.

Def. A **set of natural numbers is recursive** if its characteristic function is recursive. (The characteristic function of a set A is the characteristic function of the membership relation $\in$ for A.)

Prop. 6.29. The 2-place relation W on $\mathbb{N}$ is recursive, where $W(m, n)$ holds *iff* m is the Gödel number of a *wf* $\mathcal{A}(x_1)$ in which x_1 occurs free, and n is the Gödel number of a proof in $\mathcal{N}$ of $\mathcal{A}(0^{(m)})$.

Consequence: There is a *wf* $\mathcal{W}(x_1, x_2)$, with x_1, x_2 free, that expresses $W(m, n)$ in $\mathcal{N}$.

Def. The **Gödel sentence** $\mathcal{U}$ for $\mathcal{N}$ is the *wf* $(\forall x_2){\sim}\mathcal{W}(0^{(p)}, x_2)$, where p is the Gödel number of the *wf* $(\forall x_2){\sim}\mathcal{W}(x_1, x_2)$, in which x_1 occurs free.

Comments: $\mathcal{U}$ is interpreted in N by the following:

"For every $n \in \mathbb{N}$, $W(p, n)$ does not hold."

or

"For every $n \in \mathbb{N}$, it's not the case that (p is the Gödel number of a *wf* $\mathcal{A}(x_1)$, with x_1 free, and n is the Gödel number of a proof in $\mathcal{N}$ of $\mathcal{A}(0^{(p)})$)."

or

"For every $n \in \mathbb{N}$, n is not the Gödel number of a proof in $\mathcal{N}$ of $\mathcal{U}$."

where $\mathcal{A}(x_1)$ takes the form $(\forall x_2){\sim}\mathcal{W}(x_1, x_2)$, and $\mathcal{A}(0^{(p)})$ hence taking the form of $\mathcal{U}$.

Def. 6.30. A first order system S with $\mathcal{L}_{\mathcal{N}}$ as its language is **ω-consistent** if for any *wf* $\mathcal{A}$, with free variable x_1, for which $\vdash_{\mathcal{N}} \mathcal{A}(0^{(n)})$, for all $n \in \mathbb{N}$, then $\nvdash_{\mathcal{N}} (\exists x_1){\sim}\mathcal{A}$.

Prop. 6.31. Let S be a first order system with $\mathcal{L}_{\mathcal{N}}$ as its language. If S is ω-consistent, then S is consistent.

Prop. 6.32. (Gödel's Incompleteness Theorem)
If $\mathcal{N}$ is ω-consistent, then neither $\vdash_{\mathcal{N}} \mathcal{U}$ nor $\vdash_{\mathcal{N}} {\sim}\mathcal{U}$ (*i.e.*, if $\mathcal{N}$ is ω-consistent, then $\mathcal{N}$ is not complete).

Prop. 6.36. Let S be any extension of $\mathcal{N}$ such that the set of Gödel numbers of proper axioms of S is recursive (S is "recursively axiomatizable"). If S is consistent, then S is not complete.

Prop. (Gödel's Second Theorem)
Let S be a consistent recursively axiomatizable extension of $\mathcal{N}$. There is no proof in S of the consistency of S.