Important topics in Chapter 5 (Mathematical Systems)

Let S be a first order system.

1. **Logical axioms** of S are axioms of S that are *logically valid* (*i.e.*, (K1)-(K6)).
2. **Proper axioms** of S are axioms of S that are not logically valid.

Def. 5.3. A **first order system with equality** is an extension of K obtained by including the following *proper axioms* (axioms of equality):

(E7) $A_1^2(x_1, x_1)$

(E8) $A_1^2(t_k, u) \rightarrow A_1^2(f_i^n(t_1, ..., t_k, ..., t_n), f_i^n(t_1, ..., u, ..., t_n))$, where $t_1, ..., t_n, u$ are any terms, and f_i^n is any function letter of $\mathcal{L}$.

(E9) $(A_1^2(t_k, u) \rightarrow (A_i^n(t_1, ..., t_k, ..., t_n) \rightarrow A_i^n(t_1, ..., u, ..., t_n)))$, where $t_1, ..., t_n, u$ are any terms, and A_i^n is any predicate letter of $\mathcal{L}$.

A_1^2 is a 2-place predicate letter representing equality. *Convention*: We abbreviate $A_1^2(x, x)$ by $x = x$.

The first order system $\mathcal{N}$ of arithmetic.

The language $\mathcal{L}_{\mathcal{N}}$ of $\mathcal{N}$ contains:

individual variables:	$x_1, x_2, ...$	(below abbreviated $x, y, z, ...$)
individual constant:	a_1	(for 0)
function letters:	f_1^1, f_1^2, f_2^2	(successor $'$, sum $+$, product, $\times$)
predicate letter:	A_1^2	(equality $=$)
punctuation:	(,), ,	
connectives:	$\sim, \rightarrow$	
quantifier:	$\forall$	

$\mathcal{N}$ is the extension of K obtained by including as axioms (E7), (E8), (E9), and the following:

(N1) $(\forall x)\sim(x') = 0$

(N2) $(\forall x)(\forall y)(x' = y' \rightarrow x = y)$

(N3) $(\forall x)(x + 0 = x)$

(N4) $(\forall x)(\forall y)(x + y' = (x + y)')$

(N5) $(\forall x)(x \times 0 = 0)$

(N6) $(\forall x)(\forall y)(x \times y' = (x \times y) + x)$

(N7) $\mathcal{A}(0) \rightarrow ((\forall x)(\mathcal{A}(x) \rightarrow \mathcal{A}(x') \rightarrow (\forall x)\mathcal{A}(x))$, for each *wf* $\mathcal{A}(x)$ of $\mathcal{L}_{\mathcal{N}}$ in which x occurs free.

The first order system ZF of set theory.

The language $\mathcal{L}_{\text{ZF}}$ of ZF contains:

individual variables:	$x_1, x_2, ...$	(below abbreviated $x, y, z, w, ...$)
predicate letter:	A_1^2, A_2^2	(equality $=$; membership $\in$)
punctuation:	(,), ,	
connectives:	$\sim, \rightarrow$	(with $\leftrightarrow, \vee, \wedge$ defined as usual)
quantifier:	$\forall$	(with $\exists$ and $\exists_1$ defined as usual)

ZF is the extension of K obtained by including as axioms (E7), (E8), (E9), and the following eight:

(ZF1) $(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y))$ *(Axiom of Extensionality)*

Two sets x, y are the same *if and only if* they have the same members.

(ZF2) $(\exists x)(\forall y)\sim(y \in x)$ (*Empty Set Axiom*)

A set x exists that has no members.
Notation: (ZF1) & (ZF2) entail there is a unique empty set: Call it $\varnothing$.

(ZF3) $(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow (w = x \lor w = y))$ (*Axiom of Pairing*)

Given any sets x and y, there is a "pair" set z whose members are x and y.
Notation: (ZF1) & (ZF3) entail there is a unique pair set for any given x, y: Call it $\{x, y\}$.

(ZF4) $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists w)(w \in x \land z \in w))$ (*Axiom of Unions*)

Given any set x, there is a "union" set y whose members are all members of members of x.
Notation: (ZF1) & (ZF4) entail there is a unique union set for any set x: Call it $\cup x$.
Let $x \cup y$ represent the union set $\cup\{x, y\}$ of the pair set of x and y.

(ZF5) $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall w)(w \in z \rightarrow w \in x))$ (*Power Set Axiom*)

Given any set x, there is a "power" set y which has as members all sets whose members are members of x.
Notation: (ZF1) & (ZF4) entail there is a unique power set for any set x: Call it $\wp(x)$. Define $x \subseteq y$ ("x is a subset of y") as $(\forall z)(z \in x \rightarrow z \in y)$. Then (ZF5) can be written as $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x)$ (*i.e.*, y contains all the subsets of x).

(ZF6) $(\forall x)(\exists_1 y)\mathcal{A}(x, y) \rightarrow (\forall z)(\exists w)(\forall v)(v \in w \leftrightarrow (\exists u)(u \in z \land \mathcal{A}(u, v)))$, for every *wf* $\mathcal{A}(x, y)$ in which x and y occur free. (*Axiom Scheme of Replacement*)

Given a way $\mathcal{A}$ to relate a set x with a unique set y, then for any set z, we can form a new set w which has as its members all the sets that are related to members of z under $\mathcal{A}$.
Comment: Start with z and obtain w by replacing each member of z with its counterpart under $\mathcal{A}$.

(ZF7) $(\exists x)(\varnothing \in x \land (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$ (*Axiom of Infinity.*)

An infinite set exists.
Comment: x takes the form $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, ...\}$.

(ZF8) $(\forall x)(\sim x = \varnothing \rightarrow (\exists y)(y \in x \land \sim(\exists z)(z \in y \land z \in x)))$ (*Axiom of Foundation*)

Every non-empty set x contains a member which is disjoint from x.

Lemma: For any set x, $x \notin x$. (Consequence of ZF3 and ZF8.)

Additional axioms for formal set theory:

(AC) For any non-empty set x there is a set y which has precisely one element in common with each member of x. (*Axiom of Choice.*)

(CH) Each infinite set of real numbers either is countable or has the same cardinal number as the set of all real numbers. (*Continuum Hypothesis.*)

Comment: Two sets have the same cardinal number just when they can be put into 1-1 correspondence with each other. A set is countable just when it has the same cardinal number as $\mathbb{N}$. One can show that the cardinal number of $\mathbb{R}$ is larger than that of $\mathbb{N}$. (In fact, $\mathbb{R}$ has the same cardinal number as $\wp(\mathbb{N})$.) CH says there is no other cardinal number between that for $\mathbb{R}$ and than for $\mathbb{N}$.

Results:

1. AC and CH are consistent with ZF. (If ZF is consistent, then the extensions obtained by adding either AC or CH, or both, to ZF as axioms are consistent.)
2. Neither AC nor CH can be derived as theorems of ZF.

Skolem Paradox:
If ZF is consistent, then it has a countable model, call it M (*i.e.*, the domain D_M of M is a countable set). But under the intended interpretation, ZF describes countable and uncountable sets. How is this possible in M?

Ex. 1: For $x = \mathbb{N}$, (ZF5) is interpreted in M as referring to an uncountable set $\wp(\mathbb{N})$, all of whose members must be objects in D_M. How is this possible if D_M is countable? How can the statement "$\wp(\mathbb{N})$ is uncountable" be interpreted as true in M?

Resolution: To say "Set a is uncountable" means "There is another set b such that the members of a cannot be paired in 1-1 fashion with the members of b". (In the intended interpretation, b is $\mathbb{N}$.) And this means "Another set c does not exist whose members are the pairs of members of a and b".
Thus, statements about uncountable sets are interpreted in M as statements about whether or not certain objects in D_M (call them M-objects) exist.

Ex. 2: The statement "$\wp(\mathbb{N})$ is uncountable" is interpreted in M as a statement about certain M-objects: "There is an M-object m_1 (corresponding to $\wp(\mathbb{N})$) and there is an M-object m_2 (corresponding to $\mathbb{N}$), and there is not an M-object corresponding to the set of pairs of members of m_1 and m_2."

Comment: This resolution *relativizes* the notion of a countable set. A set can only be said to be countable or uncountable with respect to an interpretation I of ZF. To say a set x is countable in I is to say three objects exist in D_I (one for x, one for the correlate in D_I of $\mathbb{N}$, and one for the pairing set). So a set x may be countable in some interpretations, but uncountable in others.

Ex. 3: Suppose D_M is a set of objects that are countable sets in the intended interpretation of ZF (countable "real" sets). Then "$\wp(\mathbb{N})$ is uncountable" can still be interpreted in M. The M-objects now are all countable "real" sets, even the one, call it m_1, corresponding to $\wp(\mathbb{N})$. In this case m_1 is uncountable in M, but countable in the intended interpretation of ZF. (In the intended interpretation, there is $\wp(\mathbb{N})$ which is uncountable, and there is m_1 which is countable, and these are distinct. In M, there is just m_1 which plays the role of $\wp(\mathbb{N})$.)

Consequence of relativization: Which is the *real* $\wp(\mathbb{N})$? Is it the m_1 object in D_M, or is it the object we call $\wp(\mathbb{N})$ in the intended interpretation (what we think is the *real* $\wp(\mathbb{N})$)? Suppose what we think is the *real* $\wp(\mathbb{N})$ is itself a countable set, say $\mathfrak{m}_1$, with respect to an interpretation $\mathfrak{M}$ with a larger domain than the intended interpretation. In $\mathfrak{M}$, $\mathfrak{m}_1$ and the (really) real $\wp(\mathbb{N})$ are distinct. In the intended interpretation, what we take to be the really real $\wp(\mathbb{N})$ is really $\mathfrak{m}_1$.