Proposition 4.40

Preliminaries

Suppose we enlarge $\mathcal{L}$ by adding new constants b_0, b_1, .. to form $\mathcal{L}^+$. Let S be an extension of K. Now construct an extension S^+ of S by including as axioms all axioms of S and all instances of S-axioms that contain any of the new constants b_0, b_1, ... *Example*: Axiom (K5) $(\forall x_i)\mathcal{A}(x_i) \to \mathcal{A}(t)$, where t is a term free for x_i in $\mathcal{A}(x_i)$, is an axiom of S^+, as is the particular instance $(\forall x_1)A_1^1(x_1) \to A_1^1(b_1)$.

Lemma 1: If S is consistent, so is S^+.

Proof: Suppose S is consistent and S^+ is not.

Then: There's a *wf* $\mathcal{B}$ such that $\vdash_{S^+} \mathcal{B}$ and $\vdash_{S^+} (\sim\mathcal{B})$.

Note: These S^+-proofs can be converted into S-proofs. Just replace all occurrences of b-constants with a-constants that do not occur in the S^+-proofs. (There will always be such a-constants available since there is a countable infinity of them, and there can only be a finite number of *wf*s, and hence occurrences of b-constants, in any S^+-proof.)

Result: $\vdash_S \mathcal{B}$ and $\vdash_S \sim(\mathcal{B})$. But S was assumed consistent. Hence S^+ must also be consistent.

Prop. 4.40. Let S be a consistent extension of K. Then there is an interpretation of $\mathcal{L}$ in which every theorem of S is true.

Outline of Proof:

I. Enlarge $\mathcal{L}$ to $\mathcal{L}^+$ by adding new constants b_0, b_1, ... Extend S to S^+ as above. Construct a *particular* consistent extension S_∞ of S^+. Then, by Prop. 4.39, there must be a complete consistent extension of S_∞, call it T.

II. Use T to construct an interpretation I of $\mathcal{L}^+$. Prove that for every *closed wf* $\mathcal{A}$ of $\mathcal{L}^+$, $\vdash_T \mathcal{A}$ *iff* $I \vDash \mathcal{A}$.

III. Show that for *any* (open or closed) *wf* $\mathcal{B}$ of $\mathcal{L}$, if $\vdash_S \mathcal{B}$, then $I \vDash \mathcal{B}$.

Part I. Let S be a consistent extension of K. S_∞ will be the extension of S^+ that has as its axioms the union of the sets of axioms of a particular sequence of extensions S_0, S_1, ..., of S^+. This sequence is constructed in 4 steps:

1. List all *wf*s of $\mathcal{L}^+$ that contain *one* free variable: $\mathcal{F}_0(x_{i0})$, $\mathcal{F}_1(x_{i1})$, $\mathcal{F}_2(x_{i2})$, ...
2. Choose a subset $\{c_0, c_1, ...\}$ of the *b*-constants that are free for the x_{i0}, x_{i1}, ... in the list. Require:
   (i) c_0 doesn't appear in $\mathcal{F}_0(x_{i0})$.
   (ii) For $n > 0$, $c_n \notin \{c_0, ..., c_{n-1}\}$ and c_n doesn't appear in $\mathcal{F}_0(x_{i0})$, ..., $\mathcal{F}_n(x_{in})$.
3. Let $\mathcal{G}_k$ be the *wf* $\sim(\forall x_{ik})\mathcal{F}_k(x_{ik}) \rightarrow \sim\mathcal{F}_k(c_k)$.
4. Construct the sequence S_0, S_1, ... as follows:
   (i) Let $S_0 = S^+$.
   (ii) For each $n \geq 1$, let S_n be the extension of S_{n-1} obtained by adding $\mathcal{G}_{n-1}$ as a new axiom.

Lemma 2: Each of S_0, S_1, ..., is consistent.

Proof: By (weak) induction on sequence number n.

Base Step: $n = 0$. $S_0 = S^+$ is consistent (Lemma 1).

Induction Step: For $n > 0$, suppose S_n is consistent. Now show S_{n+1} is consistent.

Suppose S_{n+1} is not consistent.

Then: There's a *wf* $\mathcal{A}$ of $\mathcal{L}^+$ such that $\vdash_{Sn+1} \mathcal{A}$ and $\vdash_{Sn+1} (\sim\mathcal{A})$.

Note: $\vdash_{Sn+1} (\mathcal{A} \rightarrow (\sim\mathcal{A} \rightarrow \sim\mathcal{B}))$. (This is a tautology of L, and hence of $\mathcal{L}$. By Prop. 4.3, it is a theorem of K, and hence of the extension S_{n+1} of K.)

Thus: $\vdash_{Sn+1} (\sim\mathcal{B})$, for *any wf* $\mathcal{B}$. In particular, $\vdash_{Sn+1} (\sim\mathcal{G}_n)$. (Even though $\mathcal{G}_n$ is an axiom of S_{n+1}! This is a consequence of assuming S_{n+1} is not consistent.)

So: $\{\mathcal{G}_n\} \vdash_{Sn} (\sim\mathcal{G}_n)$. ($\vdash_{Sn+1}$ is the same as $\{\mathcal{G}_n\} \vdash_{Sn}$.)

Thus: $\vdash_{Sn} (\mathcal{G}_n \rightarrow \sim\mathcal{G}_n)$. (By the Deduction Theorem for K. $\mathcal{G}_n$ is closed so no application of Gen on a free variable in $\mathcal{G}_n$ occurs in the deduction $\{\mathcal{G}_n\} \vdash_{Sn} (\sim\mathcal{G}_n)$.)

Note: $\vdash_{Sn} ((\mathcal{A} \rightarrow \sim\mathcal{A}) \rightarrow \sim\mathcal{A})$. (Same reasoning as in above note.)

So: $\vdash_{Sn} (\sim\mathcal{G}_n)$. In other words, $\vdash_{Sn} \sim(\sim(\forall x_{in})\mathcal{F}_n(x_{in}) \rightarrow \sim\mathcal{F}_n(c_n))$.

Note: $\vdash_{Sn} (\sim(\sim\mathcal{A} \rightarrow \sim\mathcal{B}) \rightarrow \sim\mathcal{A})$ and $\vdash_{Sn} (\sim(\sim\mathcal{A} \rightarrow \sim\mathcal{B}) \rightarrow \mathcal{B})$. (Same reasoning as in first note.)

So: $\vdash_{Sn} \sim(\forall x_{in})\mathcal{F}_n(x_{in})$ and $\vdash_{Sn} \mathcal{F}_n(c_n)$.

Now: In the proof of $\mathcal{F}_n(c_n)$, we can replace all occurrences of c_n with some variable y that doesn't occur in the proof. Since c_n doesn't appear in any of the axioms of S_n used to derive $\mathcal{F}_n(c_n)$, we get a proof in S_n of $\mathcal{F}_n(y)$).

So: $\vdash_{Sn} \mathcal{F}_n(y)$.

Thus: $\vdash_{Sn} (\forall y)\mathcal{F}_n(y)$. (Gen on y.)

So: $\vdash_{Sn} (\forall x_{in})\mathcal{F}_n(x_{in})$. (Prop. 4.18.) But S_n was assumed consistent. Hence S_{n+1} must be consistent.

Lemma 3: S_∞ is consistent, for S_∞ the extension of S^+ that has as axioms all axioms of S_0, S_1, ...

Proof: Suppose S_∞ is not consistent.

Then: There's a *wf* $\mathcal{A}$ of $\mathcal{L}^+$ such that $\vdash_{S\infty} \mathcal{A}$ and $\vdash_{S\infty} (\sim\mathcal{A})$.

Note: These S_∞-proofs are finite; so they use only a finite number of axioms of S_∞. This means they are also S_n-proofs, where S_n is the member of the sequence that has as its axioms those that are used in these proofs.

Thus: $\vdash_{Sn} \mathcal{A}$ and $\vdash_{Sn} (\sim\mathcal{A})$. But S_n is consistent, for any n. Hence S_∞ must be consistent.

Since S_∞ is consistent, it has a consistent complete extension, call it T (Prop. 4.39).*

---

* Recall from the proof of Prop. 4.39 that T is constructed by again enumerating *wf*s and constructing a sequence of extensions. In this case, however, we enumerate *all wf*s of $\mathcal{L}^+$ (not just those with one free variable). And the sequence of extensions begins, in this case, with S_∞. We then go down the list of *wf*s, checking to see if each is a theorem of S_∞. If it is, we do nothing, if it isn't, we add its negation as a new axiom and get a new member of the sequence, and continue checking the list of *wf*s for theoremhood in the new extension, repeating

Part II. Use T to define an interpretation I of $\mathcal{L}^+$ as follows:

1. $D_I = \{\text{closed terms of } \mathcal{L}^+\}$†
2. ***Distinguished elements*** of D_I are the constant letters: $\overline{a}_i$ is a_i, and $\overline{b}_i$ is b_i.
3. ***Relations*** on D_I are defined by:
   $\overline{A}_i^n(d_1, ..., d_n)$ holds if $\vdash_T A_i^n(d_1, ..., d_n)$
   $\overline{A}_i^n(d_1, ..., d_n)$ does not hold if $\vdash_T \sim A_i^n(d_1, ..., d_n)$, for $d_1, ..., d_n \in D_I$.
4. ***Functions*** on D_I are defined by:
   $\overline{f}_i^n(d_1, ..., d_n) = f_i^n(d_1, ..., d_n)$, for $d_1, ..., d_n \in D_I$.

Lemma 4: For any *closed wf* $\mathcal{A}$ of $\mathcal{L}^+$, $\vdash_T \mathcal{A}$ *iff* $I \vDash \mathcal{A}$.

Proof: By induction on the number n of connectives/quantifiers in $\mathcal{A}$.

Base Step: $n = 0$, $\mathcal{A}$ is an atomic formula $A_i^n(d_1, ..., d_n)$, where $d_1, ..., d_n$ are closed terms.

1. "⇒". Supppose $\vdash_T \mathcal{A}$.
   Then: $\overline{A}_i^n(d_1, ..., d_n)$ holds in D_I. (definition of I.)
   So: For every valuation v of I, v satisfies $A_i^n(d_1, ..., d_n)$. Thus $I \vDash \mathcal{A}$.
2. "⇐". Suppose $\nvdash_T \mathcal{A}$.
   Then: $\vdash_T \sim\mathcal{A}$. (T is complete and $\mathcal{A}$ is closed.)
   So: $\overline{A}_i^n(d_1, ..., d_n)$ doesn't hold in D_I. (definition of I.)
   Thus: For every valuation v of I, v doesn't satisfy $A_i^n(d_1, ..., d_n)$. So $I \nvDash \mathcal{A}$.

Induction Step: Suppose $\mathcal{A}$ has $n > 0$ connectives/quantifiers, and for every closed *wf* $\mathcal{W}$ shorter than $\mathcal{A}$, $\vdash_T \mathcal{W}$ *iff* $I \vDash \mathcal{W}$.

Case 1: $\mathcal{A}$ has form $(\sim\mathcal{B})$, for $\mathcal{B}$ closed and shorter than $\mathcal{A}$.

1. "⇒". Supppose $\vdash_T \mathcal{A}$. (*i.e.*, $\vdash_T \sim\mathcal{B}$)
   Then: $\nvdash_T \mathcal{B}$. (T is consistent.)
   Hence: $I \nvDash \mathcal{B}$. (Inductive Hypothesis.)
   So: $I \vDash \sim\mathcal{B}$. (Cor. 3.34, $\mathcal{B}$ is closed.) Thus $I \vDash \mathcal{A}$.
2. "⇐". Suppose $I \vDash \mathcal{A}$. (*i.e.*, $I \vDash \sim\mathcal{B}$)
   Then: $I \nvDash \mathcal{B}$. (Cor. 3.34, $\mathcal{B}$ is closed.)
   So: $\nvdash_T \mathcal{B}$. (Inductive Hypothesis.)
   So: $\vdash_T \sim\mathcal{B}$. (T is complete.) Thus $\vdash_T \mathcal{A}$.

Case 2: $\mathcal{A}$ has form $(\mathcal{B} \rightarrow \mathcal{C})$, for $\mathcal{B}$, $\mathcal{C}$ closed and shorter than $\mathcal{A}$.

1. "⇒". Suppose $I \nvDash \mathcal{A}$.
   Then: $I \vDash \mathcal{B}$ and $I \vDash \sim\mathcal{C}$.
   So: $\vdash_T \mathcal{B}$ and $\nvdash_T \mathcal{C}$. (Inductive Hypothesis.)
   So: $\vdash_T \mathcal{B}$ and $\vdash_T \sim\mathcal{C}$. (T is complete.)
   Note: $\vdash_T (\mathcal{B} \rightarrow (\sim\mathcal{C} \rightarrow \sim(\mathcal{B} \rightarrow \mathcal{C})))$. (Tautology of L, hence $\mathcal{L}$. Thus theorem of T.)
   So: $\vdash_T \sim(\mathcal{B} \rightarrow \mathcal{C})$. So $\vdash_T \sim\mathcal{A}$.
   Thus: $\nvdash_T \mathcal{A}$. (T is consistent.)
2. "⇐". Suppose $\nvdash_T \mathcal{A}$.
   Then: $\vdash_T \sim\mathcal{A}$. (T is complete.) Or $\vdash_T \sim(\mathcal{B} \rightarrow \mathcal{C})$.
   Note: $\vdash_T \sim(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow \mathcal{B}$ and $\vdash_T \sim(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow \sim\mathcal{C}$. (Tautologies of L, hence theorems of T.)
   So: $\vdash_T \mathcal{B}$ and $\vdash_T \sim\mathcal{C}$.
   So: $\vdash_T \mathcal{B}$ and $\nvdash_T \mathcal{C}$. (T is consistent.)
   Hence: $I \vDash \mathcal{B}$ and $I \vDash \sim\mathcal{C}$. (Inductive Hypothesis.)
   Thus: $I \nvDash (\mathcal{B} \rightarrow \mathcal{C})$. So $I \nvDash \mathcal{A}$.

---

this process until we exhaust the list of *wf*s. T is then the extension of S_∞ that includes as axioms all axioms of sequence members.

† *Recall*: These are terms with no variables: $a_0, a_1, ..., b_0, b_1, ..., f_i^n(a_i, b_i, ...)$, *etc.*

<u>Case 3</u>: $\mathcal{A}$ has form $(\forall x_i)\mathcal{B}(x_i)$, for $\mathcal{B}(x_i)$ shorter than $\mathcal{A}$.

A. Suppose x_i does not occur free in $\mathcal{B}$.

Then: $\mathcal{B}$ is closed (since $\mathcal{A}$ is closed).
So: $\vdash_T \mathcal{B}$ *iff* $I \vDash \mathcal{B}$. (Inductive Hypothesis.)
Note: $\vdash_T \mathcal{B}$ *iff* $\vdash_T (\forall x_i)\mathcal{B}(x_i)$. (*<u>Proof</u>*: 1. "⇒": Gen on x_i. 2. "⇐": Use (K4) and MP.)
Note: $I \vDash \mathcal{B}$ *iff* $I \vDash (\forall x_i)\mathcal{B}(x_i)$. (Prop. 3.27.)
So: $\vdash_T (\forall x_i)\mathcal{B}(x_i)$ *iff* $I \vDash (\forall x_i)\mathcal{B}(x_i)$. Thus $\vdash_T \mathcal{A}$ *iff* $I \vDash \mathcal{A}$.

B. Suppose x_i occurs free in $\mathcal{B}$.

Then: x_i is the only free variable in $\mathcal{B}$ (since $\mathcal{A}$ is closed).
So: $\mathcal{B}(x_i)$ occurs in the sequence $\mathcal{F}_0(x_{i0})$, $\mathcal{F}_1(x_{i1})$, ..., say as $\mathcal{F}_m(x_{im})$.
Then: $\mathcal{A}$ has form $(\forall x_{im})\mathcal{F}_m(x_{im})$.

1. "⇐". Suppose $I \vDash \mathcal{A}$.
   Now: $\vdash_T (\forall x_{im})\mathcal{F}_m(x_{im}) \rightarrow \mathcal{F}_m(c_m)$. (K5, c_m is free for x_{im} in $\mathcal{F}_m(x_{im})$, since c_m doesn't occur in $\mathcal{F}_m(x_{im})$.)
   So: $I \vDash (\forall x_{im})\mathcal{F}_m(x_{im}) \rightarrow \mathcal{F}_m(c_m)$. (Prop. 4.4. - axioms are logically valid.)
   Hence: $I \vDash \mathcal{F}_m(c_m)$. (Prop. 3.26.)
   Thus: $\boxed{\vdash_T \mathcal{F}_m(c_m).}$ (Inductive Hypothesis.)
   Now: Suppose $\nvdash_T \mathcal{A}$.
   Then: $\vdash_T \sim\mathcal{A}$. (T is complete.) Or $\vdash_T \sim(\forall x_{im})\mathcal{F}_m(x_{im})$.
   But: $\vdash_T \sim(\forall x_{im})\mathcal{F}_m(x_{im}) \rightarrow \sim\mathcal{F}_m(c_m)$. ($\mathcal{G}_m$ is an axiom of T.)
   So: $\boxed{\vdash_T \sim\mathcal{F}_m(c_m).}$ But T is consistent. Thus it must be that $\vdash_T \mathcal{A}$.
2. "⇒". Suppose $\vdash_T \mathcal{A}$. Now suppose $I \nvDash \mathcal{A}$.
   Then: There's a valuation in I that doesn't satisfy $\mathcal{A}$.
   So: There's a valuation v that doesn't satisfy $\mathcal{F}_m(x_{im})$.
   Now: $v(x_{im}) = d$, for some closed term d in D_I.
   And: $v(d) = d$. (Valuations map constants to constants; hence closed terms to closed terms.)
   So: $v(x_{im}) = v(d)$.
   Now: We have the following:
   1. $\mathcal{F}_m(x_{im})$ is a *wf* with x_{im} free.
   2. d is a (closed) term free for x_{im} in $\mathcal{F}_m(x_{im})$.
   3. $v(x_{im}) = v(d)$.
   4. v is i-equivalent to itself.

   Thus: v satisfies $\mathcal{F}_m(d)$ *iff* v satisfies $\mathcal{F}_m(x_{im})$. (Prop. 3.23.)
   Hence: v does not satisfy $\mathcal{F}_m(d)$.
   Thus: $\boxed{I \nvDash \mathcal{F}_m(d)}$.
   Now: $\vdash_T (\forall x_{im})\mathcal{F}_m(x_{im})$. (assumption $\vdash_T \mathcal{A}$.)
   So: $\vdash_T \mathcal{F}_m(d)$. (K5, d is free for x_{im} in $\mathcal{F}_m(x_{im})$, and MP.)
   Hence: $\boxed{I \vDash \mathcal{F}_m(d)}$. (Inductive Hypothesis.) So it must be that $I \vDash \mathcal{A}$.

<u>Part III.</u>

<u>Lemma 5</u>: For *any* (open or closed) *wf* $\mathcal{B}$ of $\mathcal{L}$, if $\vdash_S \mathcal{B}$, then $I \vDash \mathcal{B}$.

<u>Proof</u>: Suppose $\vdash_S \mathcal{B}$, for some *wf* $\mathcal{B}$ of $\mathcal{L}$.
If $\mathcal{B}$ is closed, then $\vdash_T \mathcal{B}$, hence $I \vDash \mathcal{B}$. (Lemma 4: If $\mathcal{B}$ is a closed *wf* of $\mathcal{L}$, it is also a closed *wf* of $\mathcal{L}^+$.)
Suppose $\mathcal{B}$ is open.
Then: $\vdash_S \mathcal{B}'$. (Prop. 4.19, $\mathcal{B}'$ is the universal closure of $\mathcal{B}$.)
Hence: $\vdash_T \mathcal{B}'$.
Thus: $I \vDash \mathcal{B}'$. (Lemma 4.)
Hence: $I \vDash \mathcal{B}$. (Cor. 3.28.)