Important Definitions in Chapter 3 (Predicate Calculus)

Def. 3.14. An **interpretation** I of $\mathcal{L}$ consists of:
1. A non-emtpy set D_I (the domain of I).
2. A (possibly empty) collection of distiguished elements $\overline{a}_i$ of D_I.
3. A (possibly empty) collection of functions $\overline{f}_i^n$ defined on D_I.
4. A (possibly empty) collection of relations $\overline{A}_i^n$ defined on D_I.

Def. 3.17. A **valuation** in I is a function $v: \{\text{terms in } \mathcal{L}\} \to D_I$ such that

(i) $v(a_i) = \overline{a}_i$, for each constant a_i of $\mathcal{L}$

(ii) $v(x_i)$ is arbitrary, for each variable x_i of $\mathcal{L}$

(iii) $v(f_i^n(t_1, \ldots, t_n)) = \overline{f}_i^n(v(t_1), \ldots, v(t_n))$, where f_i^n is a function letter of $\mathcal{L}$ and t_1, ..., t_n are terms of $\mathcal{L}$.

Def. 3.20. Let I be an interpretation of $\mathcal{L}$ and let v be a valuation in I. Given a *wf* $\mathcal{A}$ of $\mathcal{L}$, v **satisfies** $\mathcal{A}$ just when:

(i) If $\mathcal{A}$ is atomic, $A_i^n(t_1, \ldots, t_n)$, then v satisfies $\mathcal{A}$ if $\overline{A}_i^n(v(t_1), \ldots, v(t_n))$ is true in D_I.

(ii) If $\mathcal{A}$ is $(\sim\mathcal{B})$, then v satisfies $\mathcal{A}$ if v does not satisfy $\mathcal{B}$.

(iii) If $\mathcal{A}$ is $(\mathcal{B} \to \mathcal{C})$, then v satisfies $\mathcal{A}$ if either v satisfies $(\sim\mathcal{B})$ or v satisfies $\mathcal{C}$.

(iv) If $\mathcal{A}$ is $(\forall x_i)\mathcal{B}$, then v satisfies $\mathcal{A}$ if for all valuations v' which are i-equivalent to v, v' satisfies $\mathcal{B}$.

Def. 3.19. Two valuations v, v' are ***i*-equivalent** if $v(x_j) = v'(x_j)$ for all $j \neq i$.

Def. 3.24. A *wf* $\mathcal{A}$ is **true in an interpretation *I*** if every valuation in I satisfies $\mathcal{A}$. $\mathcal{A}$ is **false in *I*** if there is no valuation in I that satisfies $\mathcal{A}$ (*i.e.*, $(\sim\mathcal{A})$ is true in I).

Notation: $I \vDash \mathcal{A}$ ("$\mathcal{A}$ is true in I" or "I semantically implies $\mathcal{A}$")

Def. Let $\mathcal{A}_0$ be a *wf* of L (statement calculus) containing statement variables p_{i_1}, ..., p_{i_k}. Let $\mathcal{A}_1$, ..., $\mathcal{A}_k$ be *wf*s of $\mathcal{L}$ (predicate calculus). Then the *wf* $\mathcal{A}$ obtained by substituting each p_{i_j} in $\mathcal{A}_0$ with $\mathcal{A}_j$ is called a **substitution instance** of $\mathcal{A}_0$.

Def. 3.30. A *wf* $\mathcal{A}$ of $\mathcal{L}$ is a **tautology of $\mathcal{L}$** if it is a substitution instance of a tautology of L.

Def. 3.32. A *wf* $\mathcal{A}$ of $\mathcal{L}$ is **closed** if it has no free occurrences of any variable.

Def. 3.35. A *wf* $\mathcal{A}$ of $\mathcal{L}$ is **logically valid** if $\mathcal{A}$ is true in all interpretations of $\mathcal{L}$. $\mathcal{A}$ is **contradictory** if it is false in all interpretations of $\mathcal{L}$.

Notation: $\vDash \mathcal{A}$ ("$\mathcal{A}$ is logically valid")

Notes:
- *Both* tautologies of $\mathcal{L}$ *and* logically valid *wf*s may be closed or open.
- If $\mathcal{A}$ is a tautology of $\mathcal{L}$, then $\mathcal{A}$ is logically valid (Prop. 3.31).
- Example of a *wf* that is logically valid but not a tautology of $\mathcal{L}$: $(\forall x_i)\mathcal{A} \to (\exists x_i)\mathcal{A}$.

Important Definitions and Propositions in Chapter 4 (Predicate Calculus)

Def. 4.2. A **proof in *K*** is a finite sequence of *wfs* $\mathcal{A}_1$, .., $\mathcal{A}_n$ of $\mathcal{L}$ such that any member is either an axiom of K or follows from previous members by MP or Gen.

If Γ is a set of *wfs* of $\mathcal{L}$, a **deduction from Γ in *K*** is a proof in K in which any member of the sequence can also be an element of Γ.

A *wf* $\mathcal{A}$ of $\mathcal{L}$ is a **theorem of *K*** if it is the last member of a proof in K.

Notation: $\Gamma \vdash_K \mathcal{A}$ ("$\mathcal{A}$ is deducible from Γ in K" or "Γ syntactically implies $\mathcal{A}$ in K")
$\vdash_K \mathcal{A}$ ("$\mathcal{A}$ is a theorem of K")

Prop. 4.5. (The Soundness Theorem for *K*)
For any *wf* $\mathcal{A}$ of $\mathcal{L}$, if $\mathcal{A}$ is a theorem of K, then $\mathcal{A}$ is logically valid. (If $\vdash_K \mathcal{A}$, then $\vDash \mathcal{A}$.)

Prop. 4.8. (The Deduction Theorem for *K*) Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be *wfs* of $\mathcal{L}$ and let Γ be a (possibly empty) set of *wfs* of L. If $\Gamma \cup \{\mathcal{A}\} \vdash_K \mathcal{B}$, and this deduction does not contain an application of Gen involving a variable that is free in $\mathcal{A}$, then $\Gamma \vdash_K (\mathcal{A} \rightarrow \mathcal{B})$.

Def. 4.34. An **extension** S of K is either K itself or a formal system obtained by adding to or modifying the axioms of K in such a way that each theorem of K is also a theorem of S.
- S is called a **first order system**.
- If S, S' are two extensions of K, S' is an extension of S just when $\{S\text{-theorems}\} \subseteq \{S'\text{-theorems}\}$.

Def. 4.36. A first order system S is **consistent** if for no *wf* $\mathcal{A}$ are both $\mathcal{A}$ and $(\sim\mathcal{A})$ theorems of S.

Def. 4.38. A first order system S is **complete** if for each closed *wf* $\mathcal{A}$, either $\vdash_S \mathcal{A}$ or $\vdash_S (\sim\mathcal{A})$.

Prop. 4.39. (Lindenbaum's Lemma for *K*) Let S be a consistent first order system. Then there is a *consistent complete* extension of S.

Prop. 4.40. Let S be a consistent extension of K. Then there is an interpretation of $\mathcal{L}$ in which every theorem of S is true.

Prop. 4.41. (The Adequacy Theorem for *K*)
If $\mathcal{A}$ is a logically valid *wf* of $\mathcal{L}$, then $\mathcal{A}$ is a theorem of K. (If $\vDash \mathcal{A}$, then $\vdash_K \mathcal{A}$.)

Def. 4.42.
(1) If Γ is a set of *wfs* in $\mathcal{L}$, then a **model of Γ** is an interpretation of $\mathcal{L}$ in which every *wf* of Γ is true.
(2) If S is a first order system, then a **model of *S*** is a model of the set of theorems of S.

Prop. 4.44. A first order system S is consistent *iff* it has a model.

Prop. 4.47. (Lowenheim-Skolem Theorem) If a first order system S has a model, then S has a model whose domain is a countable set.

Prop. 4.48. (Compactness Theorem) Let S be a first order system. If every finite subset of axioms of S has a model, then S has a model.