Chapter 32: Identity

Let R be a two-place relation and suppose a, b, c are objects that stand in it.

Def 1. R is ***transitive*** just when, if a has R to b, and b has R to c, then a has R to c.

Exs. ____is heavier than____

____is an ancestor of____

Def 2. R is ***symmetric*** just when, if a has R to b, then b has R to a.

Exs. ____is married to____

____is adjacent to____

Def 3. R is ***reflexive*** just when, for any a, a has R to a.

Exs. ____has a parent in common with____

____is as tall as____

Def 4. An ***equivalence relation*** is a relation that is transitive, symmetric, and reflexive.

Exs. ____is the same age as____

____has the same last name as____

Let A be a domain of objects and let R be an equivalence relation defined on A.

Then: (1) Every object in A is R to something (reflexivity).

Suppose: a stands in R to b and a stands in R to c.

Then: b stands in R to a (symmetry).

And: b stands in R to c (transitivity).

So: (2) If a stands in R to two things, then they stand in R to each other.

Thus: R carves up the domain into non-overlapping groups (called "***equivalence classes***") that all stand *in* R to each other.

Ex1. Domain = {U.S. citizens}

R means ____has the same last name as____

- Everyone is in an equivalence class.
- No one is in more than one equivalence class.

Ex2. Domain = {positive integers}

R means ____differs by a multiple of 3 from____

Three equivalence classes:

{0, 3, 6, 9, ...}, {2, 5, 8, 11, ...}, {1, 4, 7, 10, ...}

> ***Claim***: The identity relation is the equivalence relation that partitions a domain into the smallest equivalence classes.

1. ***Reflexivity***: For any a, $a = a$.

   > ***Ex.*** *Clark Kent* = *Clark Kent.*

2. ***Symmetry***: For any a and b, if $a = b$, then $b = a$.

   > ***Ex.*** If *Clark Kent* = *Superman*, then *Superman* = *Clark Kent.*

3. ***Transitivity***: For any a, b, c, if $a = b$, and $b = c$, then $a = c$.

   > ***Ex.*** If *Clark Kent* = *Superman*, and *Superman* = *the Superhero from Krypton*, then *Clark Kent* = *the Superhero from Krypton.*

- For any domain on which = is defined, the equivalence classes all consist of single elements.

Leibniz's Law (Indiscernibility of Identicals)

(LL) If a and b are identical, then whatever property a has, b has.

- True for the identity relation.

Identity of Indiscernibles

(IdIn) If a and b share all the same properties, then a and b are identical.

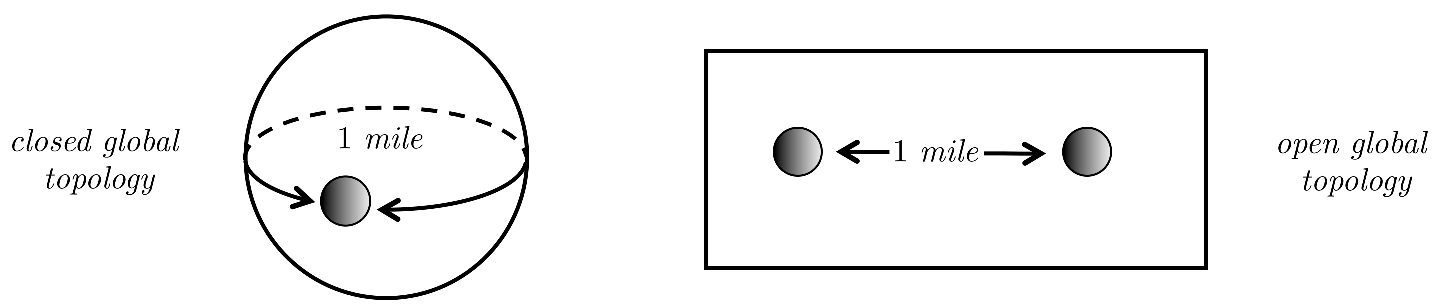
- Truth depends on the nature of the properties.

Ex1. Two peas in a pod.

- Share all monadic properties but differ on relational properties.

Ex2. Two peas in an empty universe.

- Share all monadic and relational properties, excluding spatiotemporal ones.
- If spatiotemporal properties are relational, then peas agree on these, too.
- If spatiotemporal properties are absolute, then whether or not the peas agree on them depends on the global topology of spacetime!



Chapter 33: The Language $\mathbf{QL}^=$

- *Goal*: Add identity to **QL**.
- *Convention*: Use "=" to designate the 2-place identity relation. Write "n is identical to m" as "n = m" and not "=nm"

Alphabet of ***QL***$^=$

m, n, o, ..., c_k	individual constants ($k \geq 0$)
w, x, y, z, ..., v_k	individual variables ($k \geq 0$)
A, B, C, ..., $P^0{}_k$	0-place predicates (propositional atoms) ($k \geq 0$)
F, G, H, ..., $P^1{}_k$	1-place predicates ($k \geq 0$)
L, M, =, ..., $P^2{}_k$	2-place predicates ($k \geq 0$)
⋮	
$P^n{}_k$	n-place predicates ($k \geq 0$, $n \geq 0$)
$\wedge, \vee, \neg, \supset, \forall, \exists$, (,)	connectives, quantifiers, punctuation
$\therefore, *$	argument symbols

Term of ***QL***$^{=}$

(T1$^{=}$) An individual constant or individual variable is a term of **QL**$^{=}$.

(T2$^{=}$) Nothing else is a term.

Atomic wff of ***QL***$^{=}$

(A1$^{=}$) If $\mathsf{P}^{n}{}_{k}$ is an n-place predicate symbol, $n \geq 0$, and t_1, ..., t_n are terms of **QL**$^{=}$, then $\mathsf{P}^{n}{}_{k}t_1$, ..., t_n is an atomic *wff* of **QL**$^{=}$.

(A2$^{=}$) If t_1, t_2 are terms of **QL**$^{=}$ then $t_1 = t_2$ is an atomic *wff* of **QL**$^{=}$.

(A3$^{=}$) Nothing else is an atomic *wff* of **QL**$^{=}$.

Wff of ***QL***$^{=}$

(W1$^{=}$) Any atomic *wff* of **QL**$^{=}$ is a *wff* of **QL**$^{=}$.

(W2$^{=}$) If A is a *wff* of **QL**$^{=}$, so is $\neg A$.

(W3$^{=}$) If A, B are *wffs* of **QL**$^{=}$, so is $(A \wedge B)$.

(W4$^{=}$) If A, B are *wffs* of **QL**$^{=}$, so is $(A \vee B)$.

(W5$^{=}$) If A, B are *wffs* of **QL**$^{=}$, so is $(A \supset B)$.

(W6$^{=}$) If A is a *wff* of **QL**$^{=}$ and v is a variable which occurs in A (but neither $\forall v$ nor $\exists v$ occurs in A), then $\forall vA$ is a *wff* of **QL**$^{=}$.

(W7$^{=}$) If A is a *wff* of **QL**$^{=}$ and v is a variable which occurs in A (but neither $\forall v$ nor $\exists v$ occurs in A), then $\exists vA$ is a *wff* of **QL**$^{=}$.

(W8$^{=}$) Nothing else is a *wff* of **QL**$^{=}$.

QL⁼ Semantics

A *q-valuation* on a vocabulary *V of a set of wffs of **QL**⁼*

(1) specifies a non-empty set of objects as the domain D;

(2) assigns to any constant c_k in V an object in D as its q-value;

(3) assigns a truth-value to any 0-place predicate $\mathsf{P}^0{}_k$ in V as its q-value;

(4) assigns to any n-place predicate $\mathsf{P}^n{}_k$ in V, $n > 0$, a set of n-tuples of objects $\{\langle m_1, ..., m_n\rangle, ...\}$ in D as its q-value;

*The Semantic Rules for **QL**⁼*

(Q0⁼) If A is an atomic *wff* of **QL**⁼ of the form $\mathsf{P}^n{}_k t_1, ..., t_n$, where $\mathsf{P}^n{}_k$ is an n-place predicate and $t_1, ..., t_n$ are terms of **QL**⁼, then

(a) if $n = 0$, then $A \Rightarrow_q$ T if the q-value of A is T. Otherwise $A \Rightarrow_q$ F.

(b) If $n > 0$, then $A \Rightarrow_q$ T if the n-tuple formed by taking the q-values of the terms in A in order is an element of the q-value of A. Otherwise $A \Rightarrow_q$ F.

If A is an atomic *wff* of **QL**⁼ of the form $t_1 = t_2$, where t_1, t_2 are terms of **QL**⁼, then $A \Rightarrow_q$ T if the q-values of the terms t_1 and t_2 are the same object. Otherwise $A \Rightarrow_q$ F.

(Q1$^=$) For any *wff* A, $\neg A \Rightarrow_q$ T if $A \Rightarrow_q$ F; otherwise $\neg A \Rightarrow_q$ F.

(Q2$^=$) For *wffs* A, B, $(A \wedge B) \Rightarrow_q$ T if both $A \Rightarrow_q$ T and $B \Rightarrow_q$ T; otherwise $(A \wedge B) \Rightarrow_q$ F.

(Q3$^=$) For *wffs* A, B, $(A \vee B) \Rightarrow_q$ F if both $A \Rightarrow_q$ F and $B \Rightarrow_q$ F; otherwise $(A \vee B) \Rightarrow_q$ T.

(Q4$^=$) For *wffs* A, B, $(A \supset B) \Rightarrow_q$ F if $A \Rightarrow_q$ T and $B \Rightarrow_q$ F; otherwise $(A \supset B) \Rightarrow_q$ T.

(Q5$^=$) For *wffs* A, B, $(A \equiv B) \Rightarrow_q$ T if $A \Rightarrow_q$ T and $B \Rightarrow_q$ T, or if $A \Rightarrow_q$ F and $B \Rightarrow_q$ F; otherwise $(A \equiv B) \Rightarrow_q$ F.

(Q6$^=$) For *wff* $C(...v...v...)$ with variable v free, $\forall v C(...v...v...) \Rightarrow_q$ T if $C(...v...v...) \Rightarrow_{q^+}$ T for every v-variant q^+ of q; otherwise $\forall v C(...v...v...) \Rightarrow_q$ F.

(Q7$^=$) For *wff* $C(...v...v...)$ with variable v free, $\exists v C(...v...v...) \Rightarrow_q$ T if $C(...v...v...) \Rightarrow_{q^+}$ T for at least one v-variant q^+ of q; otherwise $\exists v C(...v...v...) \Rightarrow_q$ F.

Claim: The following are q-logical truths in **QL**$^=$.

(1) $\forall x\, x = x$ (reflexivity)

(2) $\forall x \forall y (x = y \supset y = x)$ (symmetry)

(3) $\forall x \forall y \forall z ((x = y \wedge y = z) \wedge x = z)$ (transitivity)

- Is Leibniz's Law (LL) a *q*-logical truth in **QL**$^=$? (Can it be translated into a **QL**$^=$ *wff*?)

> (LL) $\forall x \forall y \forall X(x = y \supset (Xx \supset Xy))$
> where x, y range over objects and X ranges over properties.

- *But*: **QL**$^=$ is a "first-order" language.
- *Which means*: The quantifiers $\forall$, $\exists$ only range over objects, not properties.
- Formal languages that contain quantifiers that range over properties and objects are called "second-order" languages.
- *So*: Can only translate Leibniz's Law into **QL**$^=$ as a *scheme*, and not as a *wff*.

> (LS) $\forall v \forall w(v = w \supset (C(\ldots v \ldots v \ldots) \supset C(\ldots w \ldots w \ldots)))$
> where v, w are variables and $C(\ldots v \ldots v \ldots)$ is a *wff* with v free, and $C(\ldots w \ldots w \ldots)$ is a *wff* with w free.

- Instances of LS are *wffs* of **QL**$^=$.

  $\forall x \forall y(x = y \supset (Fx \supset Fy))$

  $\forall y \forall z(y = z \supset ((Lay \wedge Lby) \supset (Laz \wedge Lbz))$

- *Note*: To say all instances of LS are *q*-logical truths in **QL**$^=$ is to say identical objects *share every feature expressible in* **QL**$^=$.
- *So*: LS is weaker than LL.

Translating from English to ***QL***$^=$

a ⇒ Ahgharad		L ⇒ ____loves____	
b ⇒ Bryn		G ⇒ ____is a girl	
m ⇒ Mrs Jones			

Ex1. Angharad is none other than Mrs. Jones.

$a = m$

Ex2. Everyone except Angharad loves Bryn.

For all x, if x is not Angharad, then x loves Bryn.

For all x, if $\neg x = a$, then Lxb.

$\forall x(\neg x = a \supset Lxb)$

Ex3. Only Mrs. Jones loves Bryn.

For all x, if x is not Mrs. Jones, then x doesn't love Bryn.

For all x, if $\neg x = m$, then $\neg Lxb$.

$\forall x(\neg x = m \supset \neg Lxb)$ OR $\forall x(Lxb \supset x = m)$

Ex4. Every girl other than Angharad loves someone other than Bryn.

For all x, if x is a girl and x isn't Angharad, then x loves someone other than Bryn.

For all x, if Gx and $\neg x = a$, then $\exists y(Lxy \wedge \neg y = b)$

$\forall x((Gx \wedge \neg x = a) \supset \exists y(Lxy \wedge \neg y = b))$

Numerical Claims

1. "There is *at most* one F."

$$\forall x \forall y((Fx \wedge Fy) \supset x = y)$$

2. "There is *at least* one F."

$$\exists x Fx$$

3. "There is *exactly* one F."

$$(\forall x \forall y((Fx \wedge Fy) \supset x = y) \wedge \exists x Fx)$$

OR $$\exists x(Fx \wedge \forall y(Fy \supset y = x))$$

OR $$\exists x \forall y(Fy \equiv y = x)$$

4. "There are *at most* two Fs."

$$\forall x \forall y \forall z(((Fx \wedge Fy) \wedge Fy) \supset ((x = y \vee y = z) \vee z = x))$$

5. "There are *at least* two Fs."

$$\exists x \exists y((Fx \wedge Fy) \wedge \neg x = y)$$

6. "There are *exactly* two Fs."

$$\exists x \exists y((Fx \wedge Fy) \wedge \neg x = y) \wedge \forall z(Fz \supset (z = x \vee z = y)))$$

Let $\exists_n v \mathsf{F} v$ be shorthand for the $\mathbf{QL}^=$ *wff* that says "There are exactly n Fs".

Consider:

$$(((\exists_2 v \mathsf{F} v \wedge \exists_1 v \mathsf{G} v) \wedge \neg\exists v(\mathsf{F} v \wedge \mathsf{G} v)) \supset \exists_3 v(\mathsf{F} v \vee \mathsf{G} v))$$

"If there are 2 F s and 1 G and nothing is both an F and a G, then there are 3 things that are either F or G."

Claim: The above *wff* in $\mathbf{QL}^=$ (fully written out) is a *q*-logical truth of $\mathbf{QL}^=$.

Questions:

- How much more of natural number arithmetic is $\mathbf{QL}^=$ in disguise?
- Can natural number arithmetic be completely reduced to $\mathbf{QL}^=$?
- If so, is the formal system that results sound? Is it complete?
  - Godel: No!
- Can other branches of mathematics be competely reduced to $\mathbf{QL}^=$ (or some appropriate extention of $\mathbf{QL}^=$)?