Chapter 30: Soundness and Completeness for QL Trees

Soundness of QL Trees
If a **QL** argument is not *q*-valid, then the corresponding **QL** tree will never close.

Let: "$\vdash_{QL}$" mean "**QL** tree-entails"

Let: "$\vDash_{QL}$" mean "*q*-entails"

Then: Soundness of **QL** Trees means: If A_1, ..., $A_n \vdash_{QL} C$, then A_1, ..., $A_n \vDash_{QL} C$.

Now: Recall that a path in a tree is ***satisfiable*** if there is a valuation that makes all *wff* on the path true.

Proof of Soundness of QL Trees

Suppose: The **QL** argument A_1, ..., $A_n \therefore C$ is not *q*-valid.

Then: There's a *q*-valuation q such that $A_1 \Rightarrow_q$ T, ..., $A_n \Rightarrow_q$ T, $\neg C \Rightarrow_q$ T.

So: The initial trunk of the corresponding **QL** tree is satisfiable.

Lemma: Every possible way of extending a satisfiable path in a **QL** tree leads to *at least one* longer satisfiable path.

So: Since a satisfiable path doesn't close, a **QL** tree with a satisfiable initial trunk will never close.

Lemma: Every possible way of extending a satisfiable path in a **QL** tree leads to *at least one* longer satisfiable path.

Proof: Need to consider all possible ways of extending a satisfiable path in a **QL** tree:

Case 1: Non-branching rules (a)–(d).

If a satisfiable path is extended by a non-branching rule, it remains satisfiable.

Ex. $(A \wedge B)$
A
B

Suppose: $(A \wedge B) \Rightarrow_q$ T.
Then: $A \Rightarrow_q$ T and $B \Rightarrow_q$ T (Q2).

Case 2: Branching rules (e)–(i).

If a satisfiable path is extended by a branching rule, then at least one of the branches remains satisfiable.

Ex. $(A \vee B)$
A | B

Suppose: $(A \vee B) \Rightarrow_q$ T.
Then: Either $A \Rightarrow_q$ T or $B \Rightarrow_q$ T, or possibily both (Q3).
So: We're guaranteed at least one branch is such that q makes all *wffs* on it true.

Case 3: (¬∀) and (¬∃) rules.

If a satisfiable path is extended using (¬∀) or (¬∃), then it remains satisfiable.

Ex. ¬∀xFx
∃x¬Fx

Suppose: ¬∀xFx $\Rightarrow_q$ T.
Then: ∃x¬Fx $\Rightarrow_q$ T (V3).

Case 4: (∃) rule.

If a satisfiable path is extended using (∃), then it remains satisfiable.

Ex. ∃xFx
Fc

Suppose: ∃xFx $\Rightarrow_q$ T, and q doesn't evaluate c.
Then: There's an extension q^+ of q that does such that Fc $\Rightarrow_{q+}$ T and ∃xFx $\Rightarrow_{q+}$ T (V2).
So: The path remains satisfiable, now by q^+.

Case 5: (∀′) rule.

If a satisfiable path is extended using (∀′), then it remains satisfiable.

Ex. ∀xFx
Fc

Suppose: ∀xFx $\Rightarrow_q$ T, and q evaluates c.
Then: Fc $\Rightarrow_q$ T (V1).
Now: Suppose q doesn't evaluate c.
Then: Fc $\Rightarrow_{q+}$ T for any extension q^+ of q that does (V1).
And: ∀xFx $\Rightarrow_{q+}$ T (V5).
So: Either way, there's a valuation that satisfies the extended path.

Completeness of QL Trees

If every **QL** tree starting with the closed *wffs* A_1, ..., A_n, $\neg C$ remains open, then the **QL** argument A_1, ..., $A_n \therefore C$ is not q-valid.

Note: Completeness of **QL** Trees means: If $A_1, ..., A_n \vDash_{QL} C$, then $A_1, ..., A_n \vdash_{QL} C$.

- To prove completeness of **QL** Trees, we first need the following definitions:

Def 1. A set of *wffs* Σ is ***syntactically consistent*** if it contains no pairs of *wffs* of the form A, $\neg A$.

Def 2. A set Σ of closed **QL** *wffs* is ***saturated*** if it contains a *truth-maker* for every non-primitive *wff* in it.

- A *truth-maker* for a non-primitive *wff* A is a *wff* that must be true in order for A to be true.

<u>*So*</u>: A set Σ of closed **QL** *wffs* is *saturated* if it contains a *truth-maker* for every non-primitive *wff* in it; which means Σ is saturated if all of the following hold:

(a) If $\neg\neg A$ occurs in Σ, then so does A.
(b) If $(A \wedge B)$ occurs in Σ, then so do both A and B.
(c) If $\neg(A \vee B)$ occurs in Σ, then so do both $\neg A$ and $\neg B$.
(d) If $\neg(A \supset B)$ occurs in Σ, then so do both A and $\neg B$.
(e) If $(A \vee B)$ occurs in Σ, then so does at least one of A or B.
(f) If $\neg(A \wedge B)$ occurs in Σ, then so does at least one of $\neg A$ or $\neg B$.
(g) If $(A \supset B)$ occurs in Σ, then so does at least one of $\neg A$ or B.
(h) If $(A \equiv B)$ occurs in Σ, then so does at least one of A, B or $\neg A$, $\neg B$.
(i) If $\neg(A \equiv B)$ occurs in Σ, then so does at least one of A, $\neg B$ or $\neg A$, B.
(j) If $\neg\forall vC(\ldots v\ldots v\ldots)$ occurs in Σ, then so does $\exists v\neg C(\ldots v\ldots v\ldots)$.
(k) If $\neg\exists vC(\ldots v\ldots v\ldots)$ occurs in Σ, then so does $\forall v\neg C(\ldots v\ldots v\ldots)$.
(ℓ) If $\forall vC(\ldots v\ldots v\ldots)$ occurs in Σ, then so does $C(\ldots c\ldots c\ldots)$ for every constant c that appears in Σ, or if no constants appear in Σ, then $C(\ldots c\ldots c\ldots)$ must for some constant c.
(m) If $\exists vC(\ldots v\ldots v\ldots)$ occurs in Σ, then so does $C(\ldots c\ldots c\ldots)$ for some constant c.

<u>*Potential problem with (ℓ)*</u>

- (ℓ) specifies a truth-maker for a universal as an instance of it.
- <u>*But*</u>: If the domain contains unnamed objects, then all instances of a universal can be true, yet the universal can be false! In such cases, (ℓ) does not specify all truth-makers.
- <u>*However*</u>: (ℓ) will specify all truth-makers for a universal for a *q*-valuation that has a domain with one object for every constant in Σ and nothing else.
- Under this "chosen valuation", if all instances of a universal are true, so is the universal.

Now: Suppose the following claims are true:

(Sys) There is a systematic way to construct a **QL** tree such that either (i) it closes, or (ii) it has an open path (possibly infinite), the *wffs* on which all form a syntactically consistent, saturated set.

(Sat) Every syntactically consistent, saturated set of closed **QL** *wffs* is satisfiable.

Proof of Completeness of QL Trees

Suppose: Every **QL** tree starting with closed *wffs* A_1, ..., A_n, $\neg C$ stays open.

Then: By (Sys), there's a systematically constructible **QL** tree with an open path whose *wffs* form a syntactically consistent, saturated set.

So: By (Sat), this path is satisfiable.

So: There's a *q*-valuation that makes A_1, ..., A_n, $\neg C$ all true.

Thus: The **QL** argument A_1, ..., $A_n \therefore C$ is not *q*-valid.

Proof of (Sat). Need to show:

(C1) For any syntactically consistent, saturated set of closed **QL** *wffs* Σ, there is a *q*-valuation that makes the primitive *wffs* in Σ true.

(C2) We can choose the domain of a *q*-valuation which makes the primitives in Σ true so that it makes all the *wffs* in Σ true.

(C1) For any syntactically consistent, saturated set of closed **QL** *wffs* Σ, there is a q-valuation that makes the primitive *wffs* in Σ true.

Proof.

Suppose: Σ is a syntactically consistent, saturated set of closed **QL** *wffs*.

Now: Construct a q-valuation q in the following way (the "chosen valuation"):

(i) Domain = all numbers k such that the constant c_k is in Σ.

(ii) *Constant assignments*:
Each constant c_k is assigned the corresponding k-object.

(iii) *Predicate letter assignments*:

- Each one-place predicate letter F is assigned the set of objects k such that the *wff* Fc_k is in Σ.
- Each two-place predicate letter R is assigned the set of ordered pairs $\langle j, k\rangle$ such that the *wff* $\mathsf{Rc}_j\mathsf{c}_k$ is in Σ.
- *etc*...

Ex. $\Sigma = \{\mathsf{Fc}_5, \mathsf{Fc}_{10}, \mathsf{Rc}_{11}\mathsf{c}_{25}, \neg\mathsf{Fc}_{25}\}$, q is given by:
Domain = $\{5, 10, 11, 25\}$
$\mathsf{c}_5 \Rightarrow 5$, $\mathsf{c}_{10} \Rightarrow 10$, $\mathsf{c}_{11} \Rightarrow 11$, $\mathsf{c}_{25} \Rightarrow 25$
$\mathsf{F} \Rightarrow \{5, 10\}$, $\mathsf{R} \Rightarrow \{\langle 11, 25\rangle\}$

Now: By design, q makes all atomic *wff* in Σ true.

But: Does q make all *primitive wff* in Σ true?

Suppose: $\neg A$ is a primitive *wff* in Σ and q makes $\neg A$ false.

Then: q makes A true.

Note: A consists of a predicate letter followed by constants (since it's a *closed* primitive *wff*).

And: The objects named by the constants that occur in A must be assigned by q to the predicate letter that occurs in A. (Q0).

And: This means A must be in Σ. (This is how q assigns objects to predicate letters: each predicate letter F is assigned by q those objects i, j, k, ... such that the *wff* $\mathsf{F}\mathsf{c}_i\mathsf{c}_j\mathsf{c}_k$... is in S.)

But: A *cannot* be in Σ (since Σ is syntactically consistent).

Thus: q must make $\neg A$ true!

(C2) We can choose the domain of a q-valuation which makes the primitives in Σ true so that it makes all the *wffs* in Σ true.

Proof.

Suppose: q makes all primitive *wffs* in Σ true.

Now: Consider any non-primitive *wff* A that occurs in Σ.

Note: Since Σ is saturated, it must contain truth-makers for A, and truth-makers for the truth-makers of A; and so on...

Then: At some point in this chain of truth-makers, they become primitive *wffs*, which q makes true.

Thus: Since these initial primitive truth-makers in the chain leading to A are true, so are all the rest, including A.

So: q makes all *wff* in Σ true.

Special Case:

Suppose: A is of the form $\forall v C(...v...v...)$.

Then: Since Σ is saturated, a *wff* of the form $C(...c...c...)$ occurs in Σ for every constant c in Σ (or, if no constants occur in Σ, then $C(...c...c...)$ does, for some constant c).

Now: q has been designed so that the objects in its domain correspond 1-1 with all constants that appear in Σ.

Hence: q makes every possible instance in Σ of $\forall v C(...v...v...)$ true, and thus it makes $\forall v C(...v...v...)$ true.

(Sys) There is a systematic way to construct a **QL** tree such that either (i) it closes, or (ii) it has an open path (possibly infinite), the *wffs* on which all form a syntactically consistent, saturated set.

Proof.

*Algorithm to construct a **QL** tree such that either (i) it closes, or (ii) it has a possibily infinite open path, the wffs on which all form a syntactically consistent, saturated set:*

(1) Start at trunk and apply tree rules to each *wff*.

(2) Use (∀) to instantiate universal *wffs* with all constants that have previously appeared.

(3) Use (∃) to instantiate any existential with a new constant c, and then go back and apply (∀) again to all universals on that path with the new constant c.

Possible Results:

(a) All branches close.

(b) At least one path continues to remain open and

(i) it is finite; which means tree construction has halted; or

(ii) it is infinite; which means tree construction does not halt.

Claim: In all three cases, the *wffs* form a syntactically consistent, saturated set.

*Soundness and Completeness of **QL** Trees*

$A_1, ..., A_n \vdash_{QL} C$, if and only if $A_1, ..., A_n \vDash_{QL} C$.