Chapter 29: QL Trees

(a) $\neg\neg \mathrm{A}$ ✓
$|$
A

Add A to each open path containing $\neg\neg A$.
Check it off

(b) $(A \wedge B)$ ✓
$|$
A
B

Add A, B to each open path containing $(A \wedge B)$.
Check it off

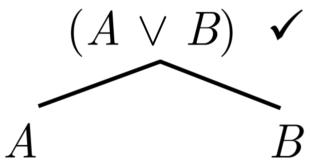
(c) $\neg(A \vee B)$ ✓
$|$
$\neg A$
$\neg B$

Add $\neg A$, $\neg B$ to each open path containing $\neg(A \vee B)$.
Check it off

(d) $\neg(A \supset B)$ ✓
$|$
A
$\neg B$

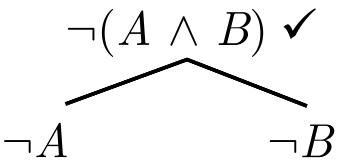
Add A, $\neg B$ to each open path containing $\neg(A \supset B)$.
Check it off

(e)



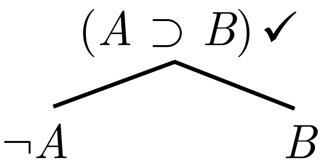
Add a fork with A, B as separate branches to each open path containing $(A \vee B)$. Check it off

(f)



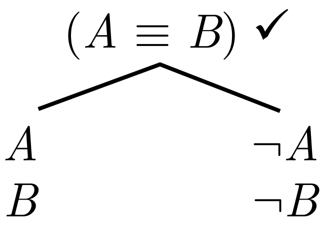
Add a fork with $\neg A$, $\neg B$ as separate branches to each open path containing $\neg(A \wedge B)$. Check it off

(g)



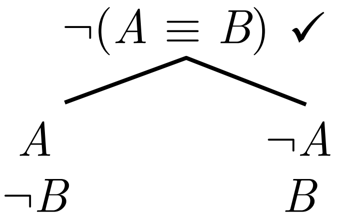
Add a fork with $\neg A$, B as separate branches to each open path containing $(A \supset B)$. Check it off

(h)



Add a fork with A, B and $\neg A$, $\neg B$ as separate branches to each open path containing $(A \equiv B)$. Check it off

(i)



Add a fork with A, $\neg B$ and $\neg A$, B as separate branches to each open path containing $\neg(A \equiv B)$. Check it off

(¬∀)

$$\begin{array}{c}\neg\forall vC \quad \checkmark \\ | \\ \exists v\neg C\end{array}$$

Add $\exists v\neg C$ to each open path containing $\neg\forall vC$. Check it off.

(¬∃)

$$\begin{array}{c}\neg\exists vC \quad \checkmark \\ | \\ \forall v\neg C\end{array}$$

Add $\forall v\neg C$ to each open path containing $\neg\exists vC$. Check it off.

(∀)

$$\begin{array}{cc}\forall vC(\ldots v\ldots v\ldots) & \\ | & \\ C(\ldots c\ldots c\ldots) & [c \text{ old}]\end{array}$$

Add $C(\ldots c\ldots c\ldots)$ to an open path containing $\forall vC(\ldots v\ldots v\ldots)$, where c is a constant on that path which hasn't already been used to instantiate $\forall vC(\ldots v\ldots v\ldots)$. ***Do not check it off.***

(∃)

$$\begin{array}{cc}\exists vC(\ldots v\ldots v\ldots) \ \checkmark & \\ | & \\ C(\ldots c\ldots c\ldots) & [c \text{ new}]\end{array}$$

Add $C(\ldots c\ldots c\ldots)$ to all open paths containing $\exists vC(\ldots v\ldots v\ldots)$, where c is a constant ***new*** to the paths. Check it off.

Advice

(1) Deal with negated quantifiers first.

(2) Instantiate existentials before universals.

Ex1. $\exists xFx, \forall x\forall y(Fy \supset \neg Lxy) \therefore \exists x\forall y\neg Lyx$

(1) $\exists xFx$ ✓

(2) $\forall x\forall y(Fy \supset \neg Lxy)$

(3) $\neg\exists x\forall y\neg Lyx$ ✓

(4) $\exists x\neg\forall y\neg Lyx$ (¬∃) on 3.

(5) Fa (∃) on 1.

(6) $\neg\forall y\neg Lya$ ✓ (∀) on 4.

(7) $\exists y\neg\neg Lya$ ✓ (¬∀) on 6.

(8) $\neg\neg Lba$ (∃) on 7.

(9) $\forall y(Fy \supset \neg Lby)$ (∀) on 2.

(10) $(Fa \supset \neg Lba)$ ✓ (∀) on 9.

(11) $\neg Fa$ $\neg Lba$ (g) on 10.

$*$ $*$

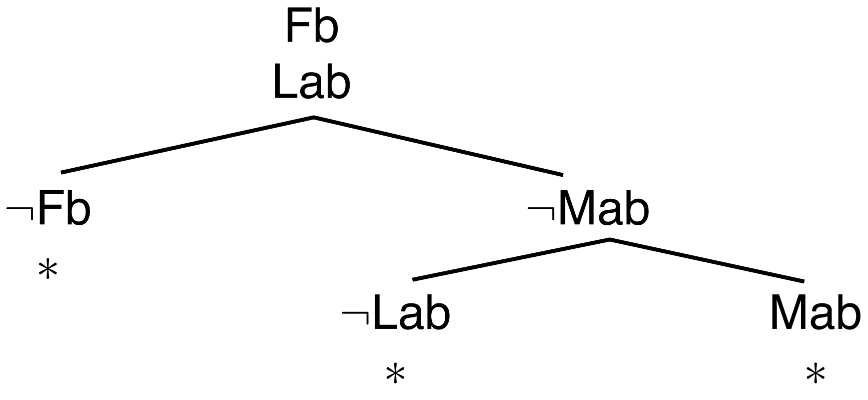
Ex2. $\forall x\exists y(Fy \wedge Lxy), \forall x\forall y(Lxy \supset Mxy) \therefore \forall x\exists y(Fy \wedge Mxy)$

(1)	$\forall x\exists y(Fy \wedge Lxy)$	
(2)	$\forall x\forall y(Lxy \supset Mxy)$	
(3)	$\neg\forall x\exists y(Fy \wedge Mxy)$ ✓	
(4)	$\exists x\neg\exists y(Fy \wedge Mxy)$ ✓	$(\neg\forall)$ on 3.
(5)	$\neg\exists y(Fy \wedge May)$ ✓	$(\exists)$ on 4.
(6)	$\forall y\neg(Fy \wedge May)$	$(\neg\exists)$ on 5.
(7)	$\exists y(Fy \wedge Lay)$ ✓	$(\forall)$ on 1.
(8)	$\forall y(Lay \supset May)$	$(\forall)$ on 2.
(9)	$(Fb \wedge Lab)$ ✓	$(\exists)$ on 7.
(10)	$\neg(Fb \wedge Mab)$ ✓	$(\forall)$ on 6.
(11)	$(Lab \supset Mab)$ ✓	$(\forall)$ on 8.

(12) (b) on 9.

(13) (f) on 10.

(14) (g) on 11.



Ex3. ∀x(Gx ⊃ ¬∃y(Fy ∧ Lxy)), (Gm ∧ ∀x(Lxm ⊃ Lmx)), Lnm ∴ ¬Fn

No girl loves any sexist pig.
Caroline is a girl who loves whoever loves her.
Henry loves Caroline.
Thus Henry isn't a sexist pig.

(1) ∀x(Gx ⊃ ¬∃y(Fy ∧ Lxy))
(2) (Gm ∧ ∀x(Lxm ⊃ Lmx)) ✓
(3) Lnm
(4) ¬¬Fn
(5) Gm — (b) on 2.
∀x(Lxm ⊃ Lmx)
(6) (Gm ⊃ ¬∃y(Fy ∧ Lmy)) ✓ — (∀) on 1.
(7) ¬Gm   ¬∃y(Fy ∧ Lmy) ✓ — (g) on 6.
*
(8) ∀y¬(Fy ∧ Lmy) — (¬∃) on 7.
(9) ¬(Fn ∧ Lmn) ✓ — (∀) on 8.
(10) ¬Fn   ¬Lmn — (f) on 9.
*
(11) (Lnm ⊃ Lmn) ✓ — (∀) on 5.
(12) ¬Lnm   Lmn — (g) on 11.
*   *

Ex4. $\forall x\forall y(\exists z Lyz \supset Lxy)$, $Lmn \therefore \forall x Lxn$

Everyone loves a lover.
Romeo loves Juliet.
Therefore everyone loves Juliet.

(1)	$\forall x\forall y(\exists z Lyz \supset Lxy)$		
(2)	Lmn		
(3)	$\neg\forall x Lxn$ ✓		
(4)	$\exists x\neg Lxn$ ✓		$(\neg\forall)$ on 3.
(5)	$\neg Lan$		$(\exists)$ on 4.
(6)	$\forall y(\exists z Lyz \supset Lay)$		$(\forall)$ on 1, a/x
(7)	$(\exists z Lnz \supset Lan)$ ✓		$(\forall)$ on 6, n/y
(8)	$\neg\exists z Lnz$ ✓	Lan	(g) on 7.
(9)	$\forall z\neg Lnz$	*	$(\neg\exists)$ on 8.
(10)	$\neg Lnm$		$(\forall)$ on 9, m/z
(11)	$\forall y(\exists z Lyz \supset Lny)$		$(\forall)$ on 1, n/x
(12)	$(\exists z Lmz \supset Lnm)$ ✓		$(\forall)$ on 11, m/y
(13)	$\neg\exists z Lmz$ ✓	Lnm	(g) on 12.
(14)	$\forall z\neg Lmz$	*	$(\neg\exists)$ on 13.
(15)	$\neg Lmn$		$(\forall)$ on 14, n/z
	*		

Claim: $\forall \mathsf{xFx} \therefore \exists \mathsf{xFx}$ is is a q-valid **QL** argument.

Proof:

Suppose: q is any q-valuation such that $\forall \mathsf{xFx} \Rightarrow_q \mathrm{T}$.

Then: *If they exist*, all x-variants q^+ of q are such that $\mathsf{Fx} \Rightarrow_{q+} \mathrm{T}$.

Now show: There *is* at least one x-variant q^+ of q.

Note: The domain of q contains at least one object (by definition); call it $\mathcal{O}$.

Now: Let q' be an x-variant of q that assigns x to the object $\mathcal{O}$.

Then: $\mathsf{Fx} \Rightarrow_{q'} \mathrm{T}$.

Hence: $\exists \mathsf{xFx} \Rightarrow_q \mathrm{T}$.

But: Is there a **QL** "tree proof" of this?

(1) $\forall \mathsf{xFx}$

(2) $\neg\exists \mathsf{xFx}$ ✓

(3) $\forall \mathsf{x}\neg \mathsf{Fx}$ $(\neg\exists)$ on 2.

- Tree construction halts!
- The $(\forall)$ Rule requires a constant that already appears on the path.

So: Unless we modify the (∀) Rule, our tree-proof system won't be complete: There will be **QL** arguments that are *q*-valid but don't have tree-proofs.

Solution:

(∀′) $\forall v C(\ldots v \ldots v \ldots)$
|
$C(\ldots c \ldots c \ldots)$ [*c* old or unprecedented]

Add $C(\ldots c \ldots c \ldots)$ to an open path containing $\forall v C(\ldots v \ldots v \ldots)$, where *c* is either a constant on that path that hasn't already been used to instantiate $\forall v C(\ldots v \ldots v \ldots)$, *or c is a new constant and there are no other constants appearing on that path.* ***Do not check it off***.

(1) $\forall x Fx$
(2) $\neg \exists x Fx$ ✓
(3) $\forall x \neg Fx$ (¬∃) on 2.
(4) Fa (∀′) on 1.
(3) $\neg Fa$ (∀′) on 3.
*

Ex5. $\forall x(Fx \supset Hx), \forall x(Gx \supset Hx) \therefore \forall x(Fx \supset Gx)$

All cats are mammals.
All dogs are mammals.
Therefore all cats are dogs.

(1) $\forall x(Fx \supset Hx)$

(2) $\forall x(Gx \supset Hx)$

(3) $\neg\forall x(Fx \supset Gx)$ ✓

(4) $\exists x\neg(Fx \supset Gx)$ ✓ $\quad$ $(\neg\forall)$ on 3.

(5) $\neg(Fa \supset Ga)$ ✓ $\quad$ $(\exists)$ on 4, a/x

(6) Fa
$\neg Ga$ $\quad$ (d) on 5.

(7) $(Fa \supset Ha)$ ✓ $\quad$ $(\forall')$ on 1, a/x

(8) $(Ga \supset Ha)$ ✓ $\quad$ $(\forall')$ on 2, a/x

(9) $\neg Fa$ $\qquad$ Ha $\quad$ (g) on 7.
$*$

(10) $\neg Ga$ $\qquad$ Ha $\quad$ (g) on 8.

- Tree construction halts!
- Can't apply $(\forall')$ Rule anymore.
- The **QL** argument is not *q*-valid.
- But we should double-check by explicitly constructing a countermodel.

- The primitive *wffs* that turned out true are {Fa, ¬Ga, Ha}.
- The corresponding vocabulary is $V = \{$a, F, G, H$\}$.

Task: Define a *q*-valuation that makes {Fa, ¬Ga, Ha} all true, and check to make sure it also makes the premises true and the conclusion false.

Q-valuation:
(1) Domain = {0}
(2) a ⇒ 0
(3) F ⇒ {0}
G ⇒ {}
H ⇒ {0}

Check:
Fa $\Rightarrow_q$ T
¬Ga $\Rightarrow_q$ T
Ha $\Rightarrow_q$ T

Claim 1: ∀x(Fx ⊃ Hx) $\Rightarrow_q$ T

Proof:

Note: There's only one *x*-variant q^+ of *q*, and it assigns x to 0.

And: Fx $\Rightarrow_{q+}$ T, since 0 is in the extension of F.

And: Hx $\Rightarrow_{q+}$ T, since 0 is in the extension of H.

So: (Fx ⊃ Hx) $\Rightarrow_{q+}$ T.

Thus: All *x*-variants of *q* make (Fx ⊃ Hx) true.

Hence: ∀x(Fx ⊃ Hx) $\Rightarrow_q$ T

- The primitive *wffs* that turned out true are $\{\mathsf{Fa}, \neg\mathsf{Ga}, \mathsf{Ha}\}$.
- The corresponding vocabulary is $V = \{\mathsf{a}, \mathsf{F}, \mathsf{G}, \mathsf{H}\}$.

Task: Define a *q*-valuation that makes $\{\mathsf{Fa}, \neg\mathsf{Ga}, \mathsf{Ha}\}$ all true, and check to make sure it also makes the premises true and the conclusion false.

Q-valuation:

(1) Domain $= \{0\}$

(2) $\mathsf{a} \Rightarrow 0$

(3) $\mathsf{F} \Rightarrow \{0\}$

$\mathsf{G} \Rightarrow \{\}$

$\mathsf{H} \Rightarrow \{0\}$

Check:

$\mathsf{Fa} \Rightarrow_q \mathrm{T}$

$\neg\mathsf{Ga} \Rightarrow_q \mathrm{T}$

$\mathsf{Ha} \Rightarrow_q \mathrm{T}$

Claim 2: $\forall\mathsf{x}(\mathsf{Gx} \supset \mathsf{Hx}) \Rightarrow_q \mathrm{T}$

Proof:

Note: There's only one x-variant q^+ of q, and it assigns x to 0.

And: $\mathsf{Gx} \Rightarrow_{q+} \mathrm{F}$, since 0 is not in the extension of G.

So: $(\mathsf{Gx} \supset \mathsf{Hx}) \Rightarrow_{q+} \mathrm{T}$.

Thus: All x-variants of q make $(\mathsf{Gx} \supset \mathsf{Hx})$ true.

Hence: $\forall\mathsf{x}(\mathsf{Gx} \supset \mathsf{Hx}) \Rightarrow_q \mathrm{T}$

- The primitive *wffs* that turned out true are $\{\mathsf{Fa}, \neg\mathsf{Ga}, \mathsf{Ha}\}$.
- The corresponding vocabulary is $V = \{\mathsf{a}, \mathsf{F}, \mathsf{G}, \mathsf{H}\}$.

Task: Define a q-valuation that makes $\{\mathsf{Fa}, \neg\mathsf{Ga}, \mathsf{Ha}\}$ all true, and check to make sure it also makes the premises true and the conclusion false.

Q-valuation:
(1) Domain $= \{0\}$
(2) $\mathsf{a} \Rightarrow 0$
(3) $\mathsf{F} \Rightarrow \{0\}$
$\mathsf{G} \Rightarrow \{\}$
$\mathsf{H} \Rightarrow \{0\}$

Check:
$\mathsf{Fa} \Rightarrow_q \mathrm{T}$
$\neg\mathsf{Ga} \Rightarrow_q \mathrm{T}$
$\mathsf{Ha} \Rightarrow_q \mathrm{T}$

Claim 3: $\forall\mathsf{x}(\mathsf{Fx} \supset \mathsf{Gx}) \Rightarrow_q \mathrm{F}$

Proof:

Note: There's only one x-variant q^+ of q, and it assigns x to 0.

And: $\mathsf{Fx} \Rightarrow_{q+} \mathrm{T}$, since 0 is in the extension of F.

And: $\mathsf{Gx} \Rightarrow_{q+} \mathrm{F}$, since 0 is not in the extension of G.

So: $(\mathsf{Fx} \supset \mathsf{Gx}) \Rightarrow_{q+} \mathrm{F}$.

Thus: Not all x-variants of q make $(\mathsf{Fx} \supset \mathsf{Gx})$ true.

Hence: $\forall\mathsf{x}(\mathsf{Gx} \supset \mathsf{Hx}) \Rightarrow_q \mathrm{F}$.