Chapter 27: Q-Valuations

The ***vocabulary*** V of a set of **QL** *wffs* is the set of constants and predicates that appear in those *wffs*.

A "*q-valuation*" for a set of *wffs* with vocabulary V does three things:

1. Fixes a domain of objects D. *Convention*: Must be non-empty, but can contain nameless objects.
2. Assigns to each constant in V, an object in D (its *reference*).
3. Assigns to each n-place predicate in V, a set of n-tuples of objects in D (its *extension*).

The ***extension*** of a predicate is the (ordered) set of objects that satisfy it.

Ex. F means ____is wise

Extension of F = {Yoda, Einstein, Socrates, Lao Tzu...}

L means ____loves____

Extension of L = {⟨Barack, Michelle⟩, ⟨Hillary, Bill⟩, ⟨Angelina, Brad⟩, ...}

An ***n-tuple*** is an ordered set of n objects.

Ex. 2-tuple: $\langle$Angelina, Brad$\rangle$

3-tuple: $\langle$Socrates, Plato, Aristotle$\rangle$

So: A q-valutation assigns to each n-place predicate, the set of n-tuples that satisfy it.

Formal Definition of q-valuation:

*A **q-valuation** on a vocabulary V of a set of wffs of **QL***

(1) specifies a non-empty set of objects as the domain D;

(2) assigns to any constant c_k in V an object in D as its q-value;

(3) assigns to any 0-place predicate $\mathsf{P}^0{}_k$ in V a truth-value as its q-value;

(4) assigns to any n-place predicate $\mathsf{P}^n{}_k$ in V, $n > 0$, a (possibly empty) set of n-tuples of objects $\{\langle m_1, ..., m_n\rangle, ...\}$ in D as its q-value.

Note: The trickiest part of the semantics of **QL** involves the semantic rules for quantifier expressions.

Recall: How should the truth-conditions for "$\forall$xFx" be set?

Suppose: $\forall$xFx is true just when all its instances in some vocabulary V are true:

$\forall$xFx is true *if and only if* (Fm $\wedge$ Fn $\wedge$ Fo $\wedge$ ...) is true.

But: If the domain has nameless objects, it's possible for (Fm $\wedge$ Fn $\wedge$ Fo $\wedge$ ...) to be true, but $\forall$xFx to be false!

So: To allow for this, consider:

$\forall$xFx is true *if and only if* (Fx is true, no matter *what* x refers to in the domain (named or nameless objects))

In general, consider:

(1) $\forall v C(\ldots v \ldots v \ldots)$ ∀xFx

(2) $\exists v C(\ldots v \ldots v \ldots)$ ∃xFx

(3) $C(\ldots v \ldots v \ldots)$ Fx

Idea: "v" stands for a generic variable. An explicit example is the variable **x**. "$C(\ldots v \ldots v \ldots)$" stands for a generic *wff* in which v occurs one or more times. An explicit example is Fx.

- (1) is true so long as (3) is true, no matter what the reference of the "pronoun" v is.
- (2) is true so long as (3) is true for at least *one* reference of the "pronoun" v.
- To implement this, need the notion of a "v-variant" of a q-valuation:

An ***extended q-valuation*** defined over vocabulary V is a q-valuation defined over V, augmented by an assignment of objects as q-values to one or more variables.

An extended q-valuation q' is a ***v-variant*** of another (extended) q-valuation q just when q' assigns some object to the variable v and otherwise agrees exactly with q.

Now stipulate:

[$\forall v C(\ldots v \ldots v \ldots)$ is true under a q-valuation q] *if and only if* [$C(\ldots v \ldots v \ldots)$ is true for all v-variants of q.]

[$\exists v C(\ldots v \ldots v \ldots)$ is true under a q-valuation q] *if and only if* [$C(\ldots v \ldots v \ldots)$ is true for at least one v-variant of q.]

*The Semantic Rules for **QL***

Let "$\Rightarrow_q$ T" mean "is true under *q*-valuation *q*".

(Q0) For atomic *wff* A

(a) If $n = 0$, then $A \Rightarrow_q$ T if the *q*-value of A is T. Otherwise $A \Rightarrow_q$ F.

(b) If $n > 0$, then $A \Rightarrow_q$ T if the n-tuple formed by taking the *q*-values of the terms in A in order is an element of the *q*-value of A. Otherwise $A \Rightarrow_q$ F.

How to unpack (Q0b):

- *Note*: An atomic *wff* A takes the form $\mathsf{P}^n{}_k t_1, \ldots, t_n$ where $\mathsf{P}^n{}_k$ is an n-place predicate and $t_1, \ldots, t_n$ are terms
- *Now*: The n-tuple formed by taking the *q*-values of the terms in A in order takes the form $\langle o_1, \ldots, o_n \rangle$, where each o_i ($i = 1, \ldots, n$) is the *q*-value of the term t_i in A.

Each of the o_i is an object in D, so $\langle o_1, \ldots, o_n \rangle$ is an n-tuple of objects in D.

- *Also*: The *q*-value of A is the set $\{\langle m_1, \ldots, m_n \rangle, \ldots\}$ of n-tuples that q assigns to A.

Each n-tuple is an n-tuple of objects in D, and the set of all these n-tuples is the extension in D of the predicate $\mathsf{P}^n{}_k$.

- *So*: An atomic *wff* of the form $\mathsf{P}^n{}_k t_1, \ldots, t_n$ for $n > 0$ says "The things refered to by the terms $t_1, \ldots, t_n$ stand in the relation $\mathsf{P}^n{}_k$".
- *And*: (Q0b) says this is true under a *q*-valuation q just when the n objects in the domain specified by q to which q assigns $t_1, \ldots, t_n$ actually do stand in the relation $\mathsf{P}^n{}_k$ to each other.

(Q1) For any *wff* A, $\neg A \Rightarrow_q \mathrm{T}$ if $A \Rightarrow_q \mathrm{F}$; otherwise $\neg A \Rightarrow_q \mathrm{F}$.

(Q2) For *wffs* A, B, $(A \wedge B) \Rightarrow_q \mathrm{T}$ if both $A \Rightarrow_q \mathrm{T}$ and $B \Rightarrow_q \mathrm{T}$; otherwise $(A \wedge B) \Rightarrow_q \mathrm{F}$.

(Q3) For *wffs* A, B, $(A \vee B) \Rightarrow_q \mathrm{F}$ if both $A \Rightarrow_q \mathrm{F}$ and $B \Rightarrow_q \mathrm{F}$; otherwise $(A \vee B) \Rightarrow_q \mathrm{T}$.

(Q4) For *wffs* A, B, $(A \supset B) \Rightarrow_q \mathrm{F}$ if $A \Rightarrow_q \mathrm{T}$ and $B \Rightarrow_q \mathrm{F}$; otherwise $(A \supset B) \Rightarrow_q \mathrm{T}$.

(Q5) For *wffs* A, B, $(A \equiv B) \Rightarrow_q \mathrm{T}$ if $A \Rightarrow_q \mathrm{T}$ and $B \Rightarrow_q \mathrm{T}$, or if $A \Rightarrow_q \mathrm{F}$ and $B \Rightarrow_q \mathrm{F}$; otherwise $(A \equiv B) \Rightarrow_q \mathrm{F}$.

(Q6) For *wff* $C(\ldots v \ldots v \ldots)$ with variable v free, $\forall v C(\ldots v \ldots v \ldots) \Rightarrow_q \mathrm{T}$ if $C(\ldots v \ldots v \ldots) \Rightarrow_{q^+} \mathrm{T}$ for every v-variant q^+ of q; otherwise $\forall v C(\ldots v \ldots v \ldots) \Rightarrow_q \mathrm{F}$.

(Q7) For *wff* $C(\ldots v \ldots v \ldots)$ with variable v free, $\exists v C(\ldots v \ldots v \ldots) \Rightarrow_q \mathrm{T}$ if $C(\ldots v \ldots v \ldots) \Rightarrow_{q^+} \mathrm{T}$ for at least one v-variant q^+ of q; otherwise $\exists v C(\ldots v \ldots v \ldots) \Rightarrow_q \mathrm{F}$.

Example: $V = \{\mathsf{m, n, F, G, L}\}$, $\{\forall \mathsf{xFx}, \forall \mathsf{y}\neg(\mathsf{Fy} \wedge \mathsf{Gy}), \forall \mathsf{x}\exists \mathsf{yLxy}\}$

To determine the truth-values of these *wffs*, first specify a *q*-valuation, call it *q*:

(1) Domain $D = \{\textit{Socrates}, \textit{Plato}, \textit{Aristotle}\}$

(2) Assignment of constants in V: $\mathsf{m} \Rightarrow \textit{Socrates}$, $\mathsf{n} \Rightarrow \textit{Plato}$

(3) Assignment of predicates in V:

$\mathsf{F} \Rightarrow$ {Socrates, Aristotle}

$\mathsf{G} \Rightarrow \{\,\}$ (empty set)

$\mathsf{L} \Rightarrow$ {⟨Socrates, Plato⟩, ⟨Plato, Aristotle⟩, ⟨Plato, Socrates⟩, ⟨Aristotle, Aristotle⟩}

Note: *D* has a "nameless" object: the Aristotle-object

Under q, what is the truth-value of $\forall \mathsf{xFx}$?

Note: $\forall \mathsf{xFx} \Rightarrow_q \mathrm{T}$ just when $\mathsf{Fx} \Rightarrow_{q+} \mathrm{T}$ for *all* x-variants q^+ of q. (Q6)

Now: There are three possible x-variants of q:

q_1: $\mathsf{x} \Rightarrow \textit{Socrates}$ q_2: $\mathsf{x} \Rightarrow \textit{Plato}$ q_3: $\mathsf{x} \Rightarrow \textit{Aristotle}$

And: $\mathsf{Fx} \Rightarrow_{q_1} \mathrm{T}$, $\mathsf{Fx} \Rightarrow_{q_2} \mathrm{F}$, $\mathsf{Fx} \Rightarrow_{q_3} \mathrm{T}$. (Q0)

Thus: Not all x-variants of q make Fx true.

Hence: $\forall \mathsf{xFx} \Rightarrow_q \mathrm{F}$

Example: $V = \{\mathsf{m, n, F, G, L}\}$, $\{\forall \mathsf{xFx}, \forall \mathsf{y}\neg(\mathsf{Fy} \wedge \mathsf{Gy}), \forall \mathsf{x}\exists \mathsf{yLxy}\}$

To determine the truth-values of these *wffs*, first specify a *q*-valuation, call it *q*:

(1) Domain $D = \{\textit{Socrates, Plato, Aristotle}\}$

(2) Assignment of constants in V: $\mathsf{m} \Rightarrow \textit{Socrates}$, $\mathsf{n} \Rightarrow \textit{Plato}$

(3) Assignment of predicates in V:

$\mathsf{F} \Rightarrow \{\text{Socrates, Aristotle}\}$

$\mathsf{G} \Rightarrow \{\,\}$ (empty set)

$\mathsf{L} \Rightarrow \{\langle\text{Socrates, Plato}\rangle, \langle\text{Plato, Aristotle}\rangle, \langle\text{Plato, Socrates}\rangle, \langle\text{Aristotle, Aristotle}\rangle\}$

Note: D has a "nameless" object: the Aristotle-object

Under q, what is the truth-value of $\forall \mathsf{y}\neg(\mathsf{Fy} \wedge \mathsf{Gy})$?

Note: $\forall \mathsf{y}\neg(\mathsf{Fy} \wedge \mathsf{Gy}) \Rightarrow_q \text{T}$ just when $\neg(\mathsf{Fy} \wedge \mathsf{Gy}) \Rightarrow_{q+} \text{T}$ for *all* *y*-variants q^+ of q. (Q6)

And: $\neg(\mathsf{Fy} \wedge \mathsf{Gy}) \Rightarrow_{q+} \text{T}$ just when $(\mathsf{Fy} \wedge \mathsf{Gy}) \Rightarrow_{q+} \text{F}$. (Q1)

And: $(\mathsf{Fy} \wedge \mathsf{Gy}) \Rightarrow_{+} \text{F}$ just when either $\mathsf{Fy} \Rightarrow_{q+} \text{F}$ or $\mathsf{Gy} \Rightarrow_{q+} \text{F}$. (Q2)

Now: For any *y*-variant q^+, $\mathsf{Gy} \Rightarrow_{q+} \text{F}$. (Since the q^+-value of G is $\{\,\}$.)

So: $\forall \mathsf{y}\neg(\mathsf{Fy} \wedge \mathsf{Gy}) \Rightarrow_q \text{T}$.

Example: $V = \{\mathsf{m, n, F, G, L}\}$, $\{\forall x Fx, \forall y \neg(Fy \wedge Gy), \forall x \exists y Lxy\}$

To determine the truth-values of these *wffs*, first specify a *q*-valuation, call it q:

(1) Domain $D = \{Socrates, Plato, Aristotle\}$

(2) Assignment of constants in V: $\mathsf{m} \Rightarrow Socrates$, $\mathsf{n} \Rightarrow Plato$

(3) Assignment of predicates in V:

$\mathsf{F} \Rightarrow$ {Socrates, Aristotle}

$\mathsf{G} \Rightarrow \{\ \}$ (empty set)

$\mathsf{L} \Rightarrow$ {⟨Socrates, Plato⟩, ⟨Plato, Aristotle⟩, ⟨Plato, Socrates⟩, ⟨Aristotle, Aristotle⟩}

Note: D has a "nameless" object: the Aristotle-object

Under q, what is the truth-value of $\forall x \exists y Lxy$?

Note: $\forall x \exists y Lxy \Rightarrow_q$ T just when $\exists y Lxy \Rightarrow_{q+}$ T for *all* x-variants q^+ of q. (Q6)

And: There are three possible x-variants of q:

q_1: $\mathsf{x} \Rightarrow Socrates$ q_2: $\mathsf{x} \Rightarrow Plato$ q_3: $\mathsf{x} \Rightarrow Aristotle$

Now: $\exists y Lxy \Rightarrow_{q+}$ T just when $\mathsf{Lxy} \Rightarrow_{q++}$ T for *at least one* y-variant q^{++} of q^+. (Q7)

And: There are three possible y-variants of q^+:

q_1': $\mathsf{y} \Rightarrow Socrates$ q_2': $\mathsf{y} \Rightarrow Plato$ q_3': $\mathsf{y} \Rightarrow Aristotle$

Now Check: For each q^+, is there at least one q^{++} such that Lxy is true?

For q_1, there's q_2'.

For q_2, there're q_1' and q_3'.

For q_3, there's q_3'.

Thus: For each q^+, there is at least one q^{++} such that $\mathsf{Lxy} \Rightarrow_{q++}$ T.

So: $\forall x \exists y Lxy \Rightarrow_q$ T.

Some results about q-valuations

(V1) *If a q-valuation q makes a universal quantification true, it makes its instances true, too:*

If a q-valuation makes $\forall v C(\ldots v \ldots v \ldots)$ true, then, given q assigns a value to c, q makes $C(\ldots c \ldots c \ldots)$ true. And if q doesn't assign a value to c, there is an extension q^+ of q which does and which makes $C(\ldots c \ldots c \ldots)$ true.

Example:

Suppose: q is a q-valuation defined for a vocabulary V that includes the constant n, and suppose $\forall \mathsf{xFx} \Rightarrow_q \mathrm{T}$.

Then: Every x-variant of q makes Fx true. (Q6).

Now: Suppose the q-value of n is the object $\mathcal{O}$.

So: The x-variant q^+ which assigns x to $\mathcal{O}$ makes Fx true.

So: $\mathcal{O}$ is in the extension of F.

So: $\mathsf{Fn} \Rightarrow_q \mathrm{T}$. (Q0)

Now suppose: V contains no constants, and $\forall \mathsf{xFx} \Rightarrow_q \mathrm{T}$.

Then: There's at least one object in the domain of q (by definition): call it $\mathcal{O}$.

Now: Define q^+ as the same as q except for assigning to $\mathcal{O}$ the new constant a.

Then: $\mathsf{Fa} \Rightarrow_{q+} \mathrm{T}$, since $\mathcal{O}$ is in the extension of F.

(V2) *If a q-valuation q makes an existential quantification true, it has an extension which makes an instance of the quantification true:*

If a q-valuation q makes $\exists v C(\ldots v \ldots v \ldots)$ true, and c is a constant not in the vocabulary of q, then there is an extension q^+ of q which gives a q-value to c and makes $C(\ldots c \ldots c \ldots)$ true.

Example:

Suppose: The vocabulary V of a q-valuation q doesn't have the constant **a**. And suppose ∃**xFx** $\Rightarrow_q$ T.

Then: There's at least one x-variant of q that makes **Fx** true. (Q7)

So: There's some object in the domain (same for q and all variants), call it $\mathcal{O}$, that can be assigned **x** so that **Fx** is true.

Now: Let q^+ be an extension of q that agrees with q and assigns **a** to $\mathcal{O}$.

Then: **Fa** $\Rightarrow_{q+}$ T.

(V3) If a q-valuation q makes $\neg\forall v C(\ldots v \ldots v \ldots)$ true, then it also makes true $\exists v \neg C(\ldots v \ldots v \ldots)$.

Example:

Suppose: A q-valuation q makes $\neg\forall$**xFx** true.

Then: $\forall$**xFx** $\Rightarrow_q$ F. (Q1)

So: Not all x-variants of q make **Fx** true. (There's at least one x-variant of q that makes **Fx** false.)

And: This x-variant makes $\neg$**Fx** true.

So: There's an x-variant of q that makes $\neg$**Fx** true.

Thus: $\exists$**x**$\neg$**Fx** $\Rightarrow_q$ T. (Q7)

(V4) If a q-valuation q makes $\neg\exists v C(\ldots v \ldots v \ldots)$ true, then it also makes true $\forall v \neg C(\ldots v \ldots v \ldots)$.

(V5) *Extending a q-valuation q to cover a new constant c doesn't affect the truth-values q assigns to wffs that do not contain c:*

Suppose a q-valuation q is defined on a vocabulary V that does not contain the constant c; and suppose q^+ is an extension of q that assigns to c some object in the domain. Let W be a *wff* using symbols in V. Then if $W \Rightarrow_q$ T, then $W \Rightarrow_{q+}$ T.