Chapter 17: Rules for PL Trees

Unpacking Rules:

(a) $\neg\neg$A
|
A

Add A to each open path containing $\neg\neg A$.

(b) $(A \wedge B)$
|
A
B

Add A, B to each open path containing $(A \wedge B)$.

(c) $\neg(A \vee B)$
|
$\neg A$
$\neg B$

Add $\neg A$, $\neg B$ to each open path containing $\neg(A \vee B)$.

(d) $(A \vee B)$
/ \
A B

Add a fork with A, B as separate branches to each open path containing $(A \vee B)$.

(e) $\neg(A \wedge B)$
/ \
$\neg A$ $\neg B$

Add a fork with $\neg A$, $\neg B$ as separate branches to each open path containing $\neg(A \wedge B)$.

Check Rule:
When a ***non-primitive wff*** has been fully unpacked, check it with the symbol "✓".

primitive wff = an atomic wff or its negation

Instructions for PL Tree Construction

(1) Start with premises and the negation of the conclusion on the trunk.

(2) Inspect each open path for an occurrance of a *wff* W and its negation $\neg W$. If these occur, close the path with the symbol "*".

(3) If there is no unchecked non-primitive *wff* on any open path, then HALT.

(4) Otherwise, unpack any unchecked non-primitive *wff* on any open path.

(5) Goto (2).

Notes:

- The Check Rule guarantees that tree construction will terminate at some point.
- The results of tree construction are:
  (a) A ***closed tree*** = every path ends with a *.
  (b) A ***completed open tree*** = every path ends with either a * or a primitive *wff*.

Basic Result (proved later):

(a) If from $A_1, A_2, ..., A_n, \neg C$ we obtain a *closed tree*, then $A_1, A_2, ..., A_n \vDash C$ (*i.e.*, the **PL** argument $A_1, A_2, ..., A_n \therefore C$ is *tautologically valid*).

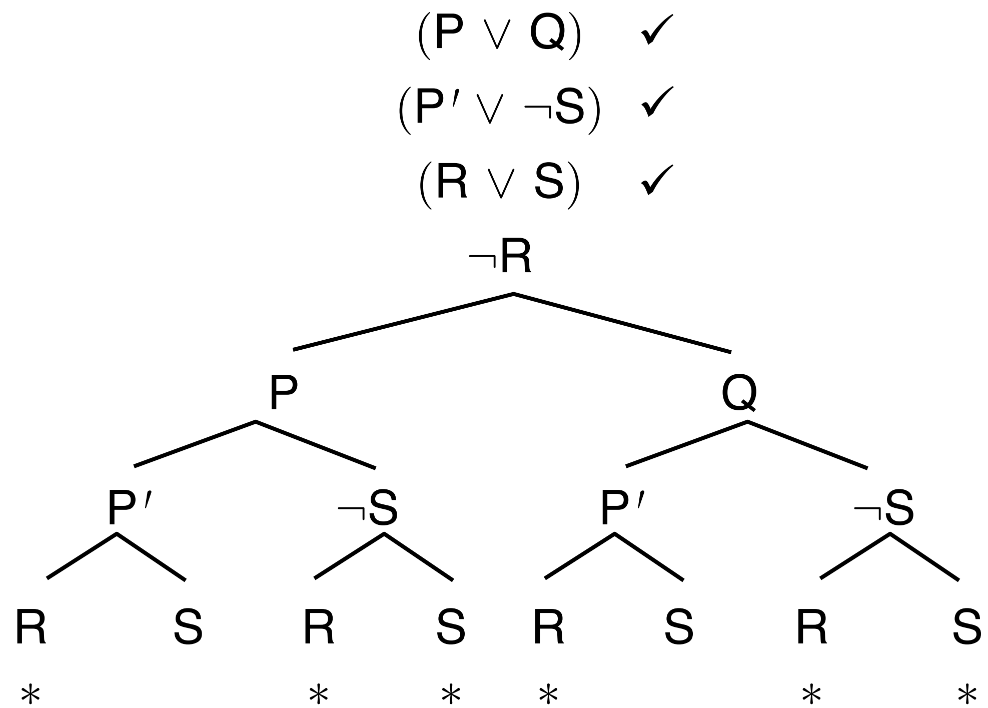
(b) If from $A_1, A_2, ..., A_n, \neg C$ we obtain a *completed open tree*, then $A_1, A_2, ..., A_n \nvDash C$ (*i.e.*, the **PL** argument $A_1, A_2, ..., A_n \therefore C$ is not *tautologically valid*).

Tree Tactics

(1) Try to apply "non-branching" rules (a), (b), (c), first, in order to reduce the number of branches.

(2) Try to close off branches as quickly as possible.

Ex.



(d) on Prem #1

(d) on Prem #2

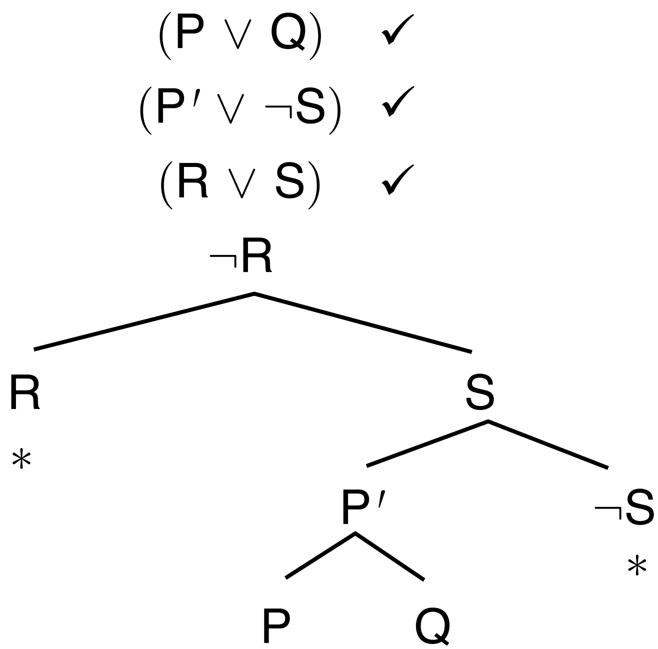
(d) on Prem #3

S $\Rightarrow$ T, P′ $\Rightarrow$ T, P $\Rightarrow$ T, R $\Rightarrow$ F, Q $\Rightarrow$ T or F

S $\Rightarrow$ T, P′ $\Rightarrow$ T, Q $\Rightarrow$ T, R $\Rightarrow$ F, P $\Rightarrow$ T or F

4 ways to make premises true and conclusion false.

Ex. Now re-do using Tree Tactics:



(d) on Prem #3

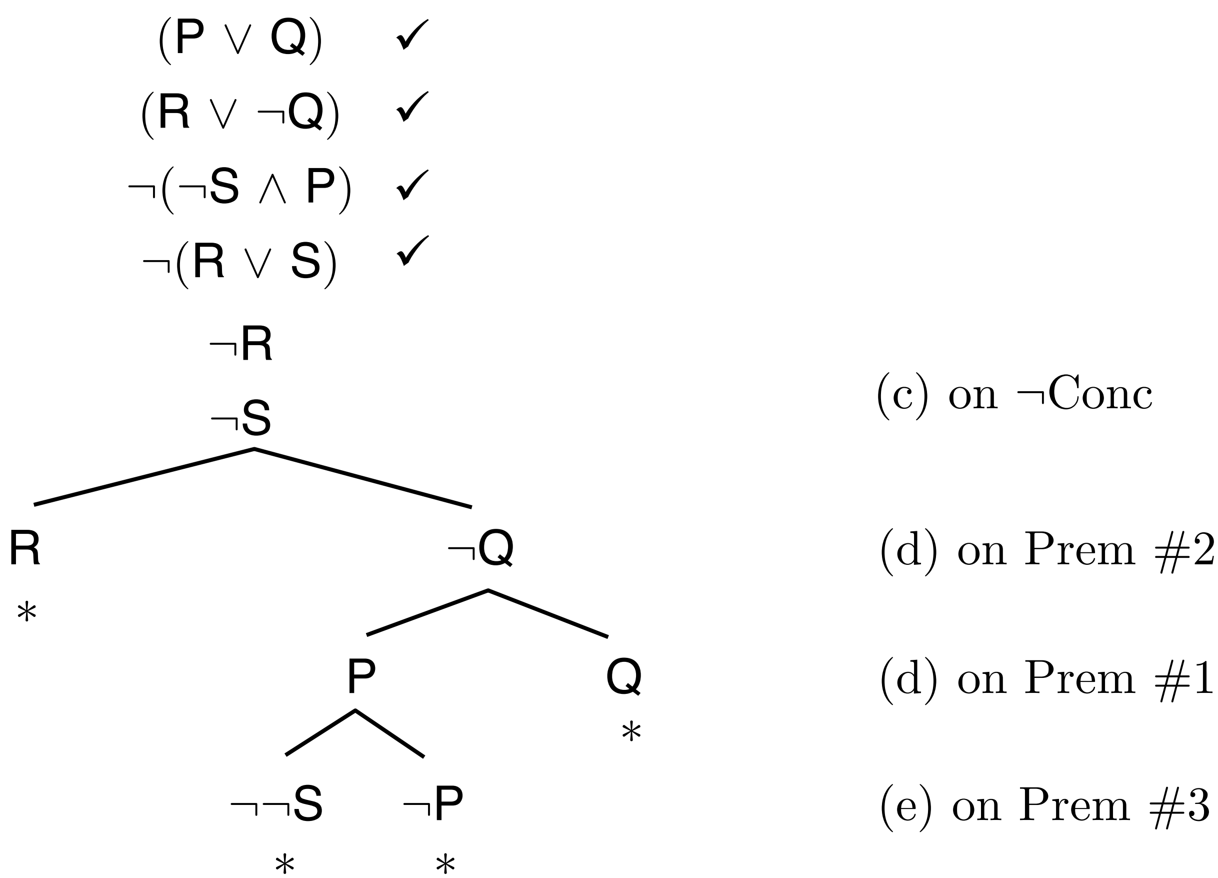
(d) on Prem #2

(d) on Prem #1

S ⇒ T, P′ ⇒ T, P ⇒ T, R ⇒ F, Q ⇒ T or F
S ⇒ T, P′ ⇒ T, Q ⇒ T, R ⇒ F, P ⇒ T or F

4 ways to make premises true and conclusion false.

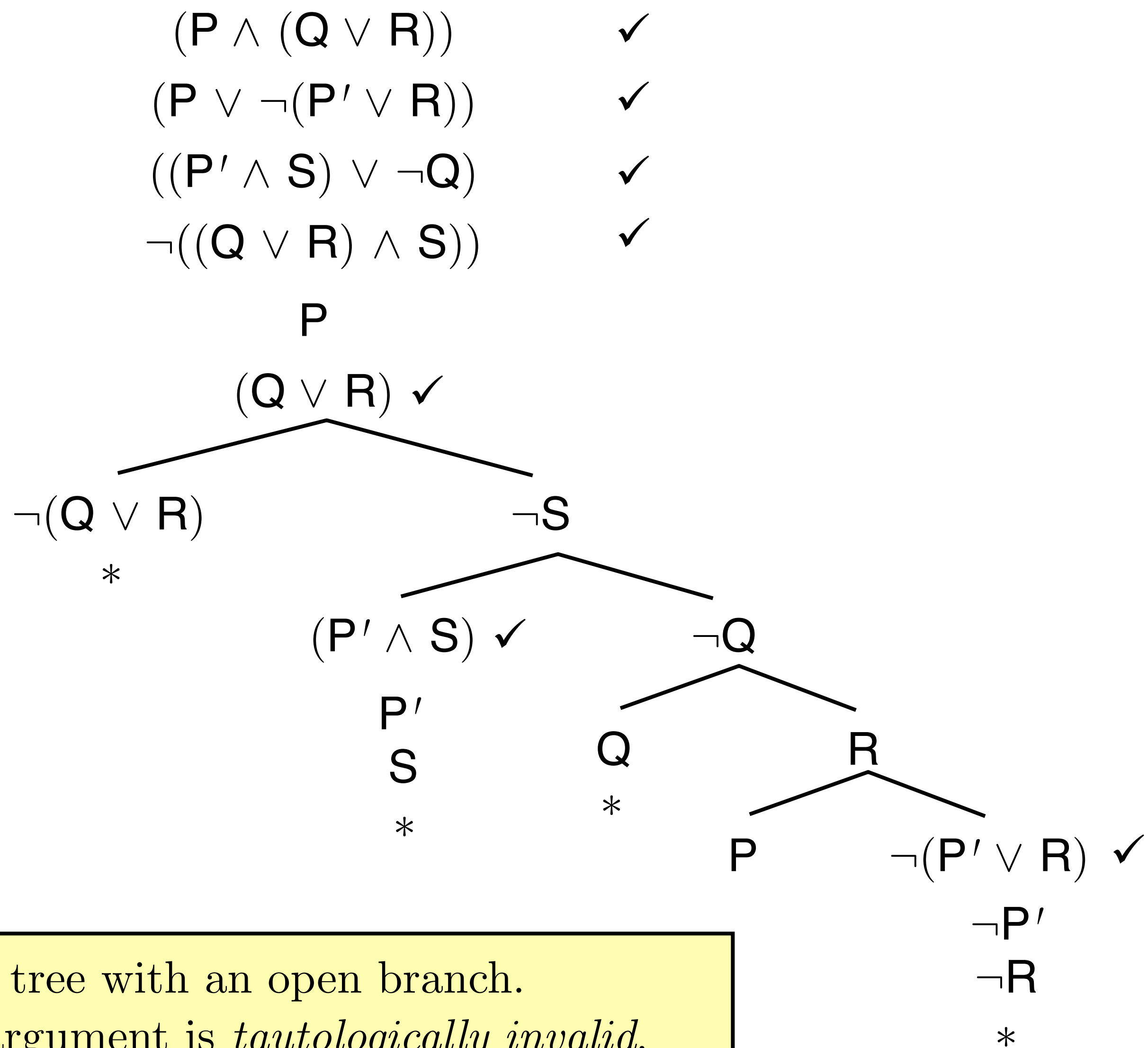
Ex1. (P ∨ Q), (R ∨ ¬Q), ¬(¬S ∧ P) ∴ (R ∨ S)



All branches close.

So: **PL** argument is *tautologically valid*.

Ex2. $(\mathsf{P} \wedge (\mathsf{Q} \vee \mathsf{R})), (\mathsf{P} \vee \neg(\mathsf{P}' \vee \mathsf{R})), ((\mathsf{P}' \wedge \mathsf{S}) \vee \neg\mathsf{Q}) \therefore ((\mathsf{Q} \vee \mathsf{R}) \wedge \mathsf{S}))$



Complete tree with an open branch.
So: **PL** argument is *tautologically invalid.*
Counterexamples when $\mathsf{P} \Rightarrow$ T, $\mathsf{P}' \Rightarrow$ T or F,
$\mathsf{Q} \Rightarrow$ F, $\mathsf{R} \Rightarrow$ T, $\mathsf{S} \Rightarrow$ F

Testing for Tautologies

Claim: If from $\neg C$ we obtain

(a) a closed tree, then $\models C$.

(b) a completed open tree, then $\not\models C$.

Ex1.

$\neg(\mathsf{P} \wedge \mathsf{Q})$ ✓

$\neg\mathsf{P}$ $\qquad\qquad$ $\neg\mathsf{Q}$

Completed open tree.

So: $\not\models (\mathsf{P} \wedge \mathsf{Q})$

Ex2. $(\neg(\neg(P \wedge Q) \wedge \neg(P \wedge R)) \vee \neg(P \wedge (Q \vee R)))$

$\{\neg[\neg(P \wedge Q) \wedge \neg(P \wedge R)] \vee \neg[P \wedge (Q \vee R)]\}$

$\neg(\neg(\neg(P \wedge Q) \wedge \neg(P \wedge R)) \vee \neg(P \wedge (Q \vee R)))$ ✓

$\neg\neg(\neg(P \wedge Q) \wedge \neg(P \wedge R))$ ✓

$\neg\neg(P \wedge (Q \vee R))$ ✓

$(\neg(P \wedge Q) \wedge \neg(P \wedge R))$ ✓

$(P \wedge (Q \vee R))$ ✓

$\neg(P \wedge Q)$ ✓

$\neg(P \wedge R)$ ✓

P

$(Q \vee R)$ ✓

$\neg P$ $\quad$ $\neg Q$

$*$

$\neg P$ $\quad$ $\neg R$

$*$

Q $\quad$ R

$*$ $\quad$ $*$

Closed tree.

So: The *wff* is a tautology.