

Chapter 16: Introducing PL Trees

Ex1. $(P \wedge \neg Q), (P' \wedge \neg Q'), (P'' \wedge \neg Q'') \therefore R$

- Truth table would contain $2^7 = 128$ rows!
- But all we really need to construct is a counterexample for which:

- (1) $(P \wedge \neg Q) \Rightarrow T$
- (2) $(P' \wedge \neg Q') \Rightarrow T$
- (3) $(P'' \wedge \neg Q'') \Rightarrow T$
- (4) $R \Rightarrow F$

- For (1), must have:

- (5) $P \Rightarrow T$
- (6) $\neg Q \Rightarrow T$ so $Q \Rightarrow F$

- For (2), (3), must have:

- (7) $P' \Rightarrow T$
- (8) $\neg Q' \Rightarrow T$ so $Q' \Rightarrow F$
- (9) $P'' \Rightarrow T$
- (10) $\neg Q'' \Rightarrow T$ so $Q'' \Rightarrow F$

Tautologically invalid!

- R nowhere occurs in premises.
- Premises are not taut. inconsistent.

Thus: The valuation $P \Rightarrow T, Q \Rightarrow F, P' \Rightarrow T, Q' \Rightarrow F, P'' \Rightarrow T, Q'' \Rightarrow F$ makes the premises true and the conclusion false.

Ex2. $(P \wedge Q), \neg(\neg R \vee S) \therefore \neg(R \wedge Q)$

- This is invalid if there is a valuation underwhich:

(1) $(P \wedge Q) \Rightarrow T$

(2) $\neg(\neg R \vee S) \Rightarrow T$

(3) $\neg(R \wedge Q) \Rightarrow F$

- For (1), must have:

(4) $P \Rightarrow T$

(5) $Q \Rightarrow T$

- For (2), must have:

(6) $(\neg R \vee S) \Rightarrow F$

(7) $\neg R \Rightarrow F$

(8) $S \Rightarrow F$

(9) $R \Rightarrow T$

- For (3), must have:

(10) $(R \wedge Q) \Rightarrow T$

(11) $R \Rightarrow T$

(12) $Q \Rightarrow T$

Thus: The valuation $P \Rightarrow T, Q \Rightarrow T, R \Rightarrow T, S \Rightarrow F$
makes the premises true and the conclusion false.

So: **PL** arugment is *tautologically invalid*.

Ex3. $(P \wedge \neg Q) \therefore \neg(Q \wedge R)$

Require:

(1) $(P \wedge \neg Q) \Rightarrow T$

(2) $\neg(Q \wedge R) \Rightarrow F$

So:

(3) $P \Rightarrow T$

(4) $\neg Q \Rightarrow T$

(5) $Q \Rightarrow F$

And:

(6) $(Q \wedge R) \Rightarrow T$

(7) $Q \Rightarrow T$

(8) $R \Rightarrow T$

But this can't be!

So: It's not possible to construct a valuation that makes premises true and conclusion false.

So: **PL** argument is *tautologically valid!*

Ex4. (Branching case) $(P \vee Q) \therefore P$

(1) $(P \vee Q) \Rightarrow T$

(2) $P \Rightarrow F$

(3) $P \Rightarrow T$ $Q \Rightarrow T$
 * *

Right branch is possible.

So: Under the valuation $P \Rightarrow F, Q \Rightarrow T$, the premises are true and the conclusion is false.

So: **PL** argument is *tautologically invalid*.

Left branch is not possible.

Convention: Mark it with "*".

Ex5. $(P \vee Q), \neg P \therefore Q$

(1) $(P \vee Q) \Rightarrow T$

(2) $\neg P \Rightarrow T$

(3) $Q \Rightarrow F$

(4) $P \Rightarrow T$ $Q \Rightarrow T$
 * *

*All branches end in *.*

Thus: No valuation exists that makes premises true and conclusion false.

So: **PL** argument is *tautologically valid*.

Ex6. $\neg(P \wedge Q), (P \wedge R) \therefore \neg(R \vee Q)$

(1) $\neg(P \wedge Q) \Rightarrow T$

(2) $(P \wedge R) \Rightarrow T$

(3) $\neg(R \vee Q) \Rightarrow F$

(4) $(P \wedge Q) \Rightarrow F$ (from 1)

(5) $P \Rightarrow T$ (from 2)

(6) $R \Rightarrow T$ (from 2)

(7) $(R \vee Q) \Rightarrow T$ (from 3)

(8) $P \Rightarrow F$ $Q \Rightarrow F$ (from 4)

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(9) $R \Rightarrow T$ $Q \Rightarrow T$ (from 7)

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The valuation $P \Rightarrow T, Q \Rightarrow F, R \Rightarrow T$ makes premises true and conclusion false.

So: PL argument is *tautologically invalid*.

"T"-only Trees

Trick 1: Start by assigning T to premises and T to the *negation* of conclusion.

Trick 2: For any *wff* C , replace occurrences of $C \Rightarrow F$ with $\neg C \Rightarrow T$.

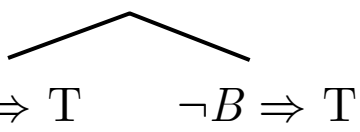
Trick 2 may require skipping steps in a tree:

Skipped steps:

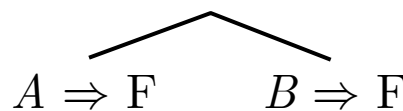
Ex. (n) $\neg(A \vee B) \Rightarrow T$
(n+1) $\neg A \Rightarrow T$
(n+2) $\neg B \Rightarrow T$

$(A \vee B) \Rightarrow F$
 $A \Rightarrow F$
 $B \Rightarrow F$

(m) $\neg(A \wedge B) \Rightarrow T$
(m+1) $\neg A \Rightarrow T$ $\neg B \Rightarrow T$



Skipped steps:
 $(A \wedge B) \Rightarrow F$
 $A \Rightarrow F$ $B \Rightarrow F$



Ex6 again: $\neg(P \wedge Q), (P \wedge R) \therefore \neg(R \vee Q)$

(1) $\neg(P \wedge Q) \Rightarrow T$

(2) $(P \wedge R) \Rightarrow T$

(3) $\neg\neg(R \vee Q) \Rightarrow T$

(4) $P \Rightarrow T$ (from 2)

(5) $R \Rightarrow T$ (from 2)

(6) $(R \vee Q) \Rightarrow T$ (from 3)

(7) $\neg P \Rightarrow T$ $\neg Q \Rightarrow T$ (from 1, skipping steps!)

(9) $R \Rightarrow T$ $Q \Rightarrow T$ (from 6)

*

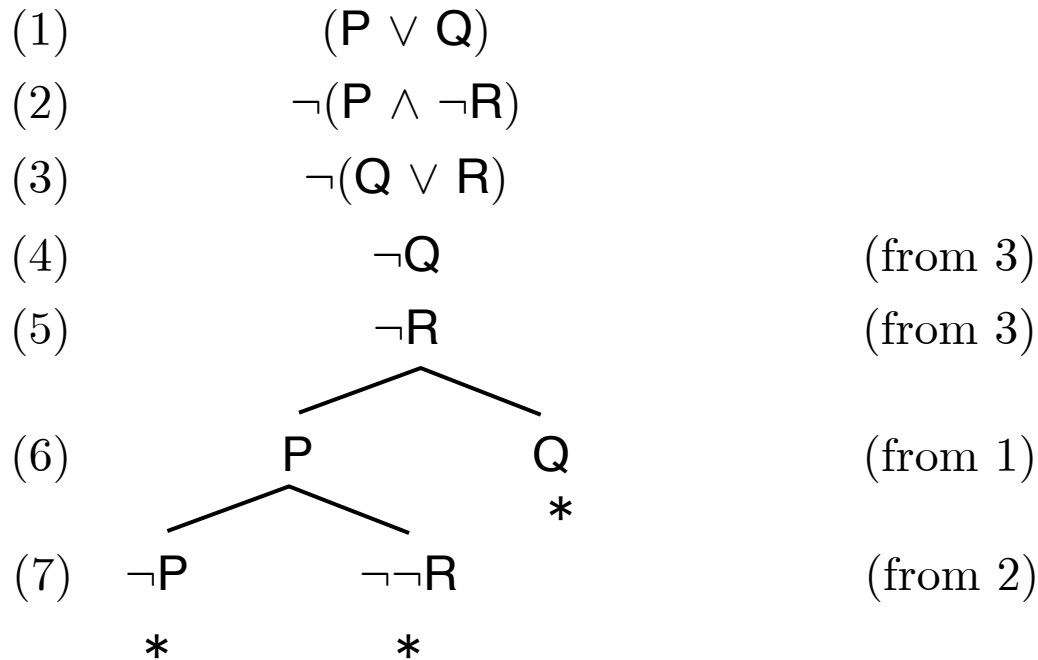
The valuation $P \Rightarrow T, Q \Rightarrow F, R \Rightarrow T$ makes premises true and conclusion false.

So: **PL** argument is *tautologically invalid*.

"Unsigned" Trees

In "T-only" trees, delete all occurrences of the symbols " \Rightarrow " and "T".

Ex7: $(P \vee Q), \neg(P \wedge \neg R) \therefore (Q \vee R)$



All branches close up!

So: The **PL** arugment is tautologically invalid.

Ex8: $(P \vee (Q \wedge R)), (\neg P \vee R), \neg(Q \vee \neg\neg S) \therefore (S \wedge R)$

- (1) $(P \vee (Q \wedge R))$
 (2) $(\neg P \vee R)$
 (3) $\neg(Q \vee \neg\neg S)$
 (4) $\neg(S \wedge R)$
 (5) $\neg Q$ from (3)
 (6) $\neg\neg\neg S$ from (3)
 (7) $\neg S$ from (6)
- (8) $\begin{array}{cc} P & (Q \wedge R) \end{array}$ from (1)
- (9) $\begin{array}{ccc} \neg P & R & R \\ * & & \end{array}$ from (8) and (2)
- (10) $\begin{array}{ccc} \neg S & \neg R & Q \\ & * & * \end{array}$ from (8) and (4)

The valuation $S \Rightarrow F, R \Rightarrow T, P \Rightarrow T, Q \Rightarrow F$ makes premises true and conclusion false.

So: The **PL** arugment is tautologically invalid.