

# Chapter 16: Introducing PL Trees

Ex1.  $(P \wedge \neg Q), (P' \wedge \neg Q'), (P'' \wedge \neg Q'') \therefore R$

- Truth table would contain  $2^7 = 128$  rows!
- But all we really need to construct is a counterexample for which:

- (1)  $(P \wedge \neg Q) \Rightarrow T$
- (2)  $(P' \wedge \neg Q') \Rightarrow T$
- (3)  $(P'' \wedge \neg Q'') \Rightarrow T$
- (4)  $R \Rightarrow F$

- For (1), must have:

- (5)  $P \Rightarrow T$
- (6)  $\neg Q \Rightarrow T$  so  $Q \Rightarrow F$

- For (2), (3), must have:

- (7)  $P' \Rightarrow T$
- (8)  $\neg Q' \Rightarrow T$  so  $Q' \Rightarrow F$
- (9)  $P'' \Rightarrow T$
- (10)  $\neg Q'' \Rightarrow T$  so  $Q'' \Rightarrow F$

***Tautologically invalid!***

- $R$  nowhere occurs in premises.
- Premises are not taut. inconsistent.

*Thus:* The valuation  $P \Rightarrow T, Q \Rightarrow F, P' \Rightarrow T, Q' \Rightarrow F, P'' \Rightarrow T, Q'' \Rightarrow F$  makes the premises true and the conclusion false.

Ex2.  $(P \wedge Q), \neg(\neg R \vee S) \therefore \neg(R \wedge Q)$

- This is invalid if there is a valuation underwhich:

- (1)  $(P \wedge Q) \Rightarrow T$
- (2)  $\neg(\neg R \vee S) \Rightarrow T$
- (3)  $\neg(R \wedge Q) \Rightarrow F$

- For (1), must have:

- (4)  $P \Rightarrow T$
- (5)  $Q \Rightarrow T$

- For (2), must have:

- (6)  $(\neg R \vee S) \Rightarrow F$
- (7)  $\neg R \Rightarrow F$
- (8)  $S \Rightarrow F$
- (9)  $R \Rightarrow T$

- For (3), must have:

- (10)  $(R \wedge Q) \Rightarrow T$
- (11)  $R \Rightarrow T$
- (12)  $Q \Rightarrow T$

Thus: The valuation  $P \Rightarrow T, Q \Rightarrow T, R \Rightarrow T, S \Rightarrow F$  makes the premises true and the conclusion false.  
So: PL arugment is *tautologically invalid*.

Ex3.  $(P \wedge \neg Q) \therefore \neg(Q \wedge R)$

Require:

- (1)  $(P \wedge \neg Q) \Rightarrow T$
- (2)  $\neg(Q \wedge R) \Rightarrow F$

So:

- (3)  $P \Rightarrow T$
- (4)  $\neg Q \Rightarrow T$
- (5)  $Q \Rightarrow F$

And:

- (6)  $(Q \wedge R) \Rightarrow T$
- (7)  $Q \Rightarrow T$
- (8)  $R \Rightarrow T$

*But this can't be!*

So: It's not possible to construct a valuation that makes premises true and conclusion false.

So: PL argument is ***tautologically valid!***

Ex4. (Branching case)  $(P \vee Q) \therefore P$

(1)  $(P \vee Q) \Rightarrow T$

(2)  $P \Rightarrow F$

(3) 
$$\begin{array}{ccc} & \swarrow & \searrow \\ P \Rightarrow T & & Q \Rightarrow T \\ * & & \end{array}$$

*Right branch is possible.*

So: Under the valuation  $P \Rightarrow F, Q \Rightarrow T$ , the premises are true and the conclusion is false.

So: **PL** argument is *tautologically invalid*.

*Left branch is not possible.*

Convention: Mark it with "\*".

Ex5.  $(P \vee Q), \neg P \therefore Q$

(1)  $(P \vee Q) \Rightarrow T$

(2)  $\neg P \Rightarrow T$

(3)  $Q \Rightarrow F$

(4) 
$$\begin{array}{ccc} & \swarrow & \searrow \\ P \Rightarrow T & & Q \Rightarrow T \\ * & & * \end{array}$$

*All branches end in \*.*

Thus: No valuation exists that makes premises true and conclusion false.

So: **PL** argument is *tautologically valid*.

Ex6.  $\neg(P \wedge Q), (P \wedge R) \therefore \neg(R \vee Q)$

- (1)  $\neg(P \wedge Q) \Rightarrow T$
- (2)  $(P \wedge R) \Rightarrow T$
- (3)  $\neg(R \vee Q) \Rightarrow F$
- (4)  $(P \wedge Q) \Rightarrow F$  (from 1)
- (5)  $P \Rightarrow T$  (from 2)
- (6)  $R \Rightarrow T$  (from 2)
- (7)  $(R \vee Q) \Rightarrow T$  (from 3)
- (8) 
$$\begin{array}{c} P \Rightarrow F & Q \Rightarrow F \\ * & \begin{array}{c} R \Rightarrow T & Q \Rightarrow T \\ * \end{array} \end{array} \quad (from \ 4)$$
- (9)  $R \Rightarrow T \quad Q \Rightarrow T \quad (from \ 7)$

The valuation  $P \Rightarrow T, Q \Rightarrow F, R \Rightarrow T$  makes premises true and conclusion false.

So: **PL** argument is *tautologically invalid*.

## "T"-only Trees

Trick 1: Start by assigning T to premises and F to the *negation* of conclusion.

Trick 2: For any wff  $C$ , replace occurrences of  $C \Rightarrow F$  with  $\neg C \Rightarrow T$ .

Trick 2 may require skipping steps in a tree:

Ex. (n)  $\neg(A \vee B) \Rightarrow T$

(n+1)  $\neg A \Rightarrow T$

(n+2)  $\neg B \Rightarrow T$

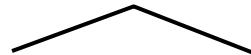
Skipped steps:

$(A \vee B) \Rightarrow F$

$A \Rightarrow F$

$B \Rightarrow F$

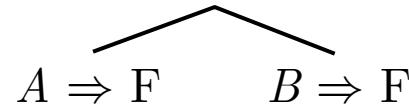
(m)  $\neg(A \wedge B) \Rightarrow T$



(m+1)  $\neg A \Rightarrow T$      $\neg B \Rightarrow T$

Skipped steps:

$(A \wedge B) \Rightarrow F$



Ex6 again:  $\neg(P \wedge Q), (P \wedge R) \therefore \neg(R \vee Q)$

- (1)  $\neg(P \wedge Q) \Rightarrow T$
- (2)  $(P \wedge R) \Rightarrow T$
- (3)  $\neg\neg(R \vee Q) \Rightarrow T$
- (4)  $P \Rightarrow T$  (from 2)
- (5)  $R \Rightarrow T$  (from 2)
- (6)  $(R \vee Q) \Rightarrow T$  (from 3)

(7)  $\neg P \Rightarrow T$     $\neg Q \Rightarrow T$  (from 1, skipping steps!)

\*

(9)  $R \Rightarrow T$     $Q \Rightarrow T$  (from 6)

\*

The valuation  $P \Rightarrow T, Q \Rightarrow F, R \Rightarrow T$  makes premises true and conclusion false.

So: **PL** argument is *tautologically invalid*.

## "Unsigned" Trees

In "T-only" trees, delete all occurrences of the symbols " $\Rightarrow$ " and "T".

Ex7:  $(P \vee Q), \neg(P \wedge \neg R) \therefore (Q \vee R)$

(1)  $(P \vee Q)$

(2)  $\neg(P \wedge \neg R)$

(3)  $\neg(Q \vee R)$

(4)  $\neg Q$  (from 3)

(5)  $\neg R$  (from 3)

(6) (from 1)

(7) (from 2)

*All branches close up!*

So: The **PL** argument is tautologically invalid.

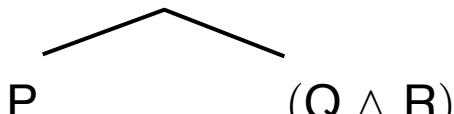
Ex8:  $(P \vee (Q \wedge R)), (\neg P \vee R), \neg(Q \vee \neg \neg S) \therefore (S \wedge R)$

- (1)  $(P \vee (Q \wedge R))$
- (2)  $(\neg P \vee R)$
- (3)  $\neg(Q \vee \neg \neg S)$
- (4)  $\neg(S \wedge R)$

(5)  $\neg Q$  from (3)

(6)  $\neg \neg \neg S$  from (3)

(7)  $\neg S$  from (6)

(8)  from (1)

(9)  from (8) and (2)

(10)  from (8) and (4)

The valuation  $S \Rightarrow F, R \Rightarrow T, P \Rightarrow T, Q \Rightarrow F$  makes premises true and conclusion false.

So: The **PL** argument is tautologically invalid.