Chapter 14: The Language of PLC

Motivation:

_Ex1_. (1) If Jack bet on the Cardinals, then Jack lost his money.

(2) Jack did bet on the Cardinals. **_VALID_**

(3) So Jack lost his money.

(1) If Jack bet on the Cardinals, then Jack lost his money.

(2) Jack lost his money. **_INVALID_**

(3) So Jack must have bet on the Cardinals.

_Ex2_. (1) If Jack bet on the Cardinals, then Jack lost his money.

(2) Jack did not lose his money. **_VALID_**

(3) So Jack didn't bet on the Cardinals.

(1) If Jack bet on the Cardinals, then Jack lost his money.

(2) Jack did not bet on the Cardinals. **_INVALID_**

(3) So Jack did not lose his money.

Given a ***conditional***, if A then C, A is called the ***antecedent*** and B is called the ***consequent***.

An inference of the general form

(1) If A then C.

(2) A.

(3) So C.

is called ***modus ponens***.

An inference of the general form

(1) If A then C.

(2) Not C.

(3) So not A.

is called ***modus tollens***.

- Define a truth-function to represent conditional propositions.
- Use the symbol "$\supset$" to represent the connective "if...then".

Semantic Rule for $\supset$

For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ F, then $(A \supset B) \Rightarrow$ F. Otherwise $(A \supset B) \Rightarrow$ T.

A	B	$(A \supset B)$
T	T	T
T	F	F
F	T	T
F	F	T

A	C	$(A \supset C)$			
T	T	T			
T	F	F			
F	T	T	T	F	F
F	F	T	F	T	F
		(a)	(b)	(c)	(d)

- (b) is same as C.

  <u>*But*</u>: $(A \supset C)$ should be different from just C.

- (c) would mean $(A \supset C)$ is same as $(C \supset A)$.

  <u>*But*</u>: This is not always the case! Let A = Jo is a man, C = Jo is a human.

- (d) is same as $\wedge$.

Does adopting (a) "work"?

Modus ponens in **PL**: $(P \supset Q)$, $P \therefore Q$

P	Q	(P ⊃ Q)	P	Q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

No row in which the premises are all T and the conclusion is F.
So: **PL** argument is *tautologically valid*.

Modus tollens in **PL**: $(P \supset Q)$, $\neg Q \therefore \neg P$

P	Q	(P ⊃ Q)	¬Q	¬P
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

No row in which the premises are all T and the conclusion is F.
So: **PL** argument is *tautologically valid*.

"Only if" sentences in English

"A only if C" means "If A then C"

So: "A only if C" is translated into **PL** as $(A \supset C)$.

The sentence after the "only if" is the consequent in a conditional.

Ex. "There is fire only if there is oxygen."

means

"If there is fire, then there is oxygen."

and not

"If there is oxygen, then there is fire."

"If and only if" sentences in English

"A if and only if C" means "If A then C and if C then A"

So: "A if and only if C" is translated into **PL** as $((A \supset C) \wedge (C \supset A))$.

Convention

- Introduce another connective for "if and only if".
- Call it the ***biconditional*** "$\equiv$".

Semantic Rule for $\equiv$

For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ T, or if $A \Rightarrow$ F and $B \Rightarrow$ F, then $(A \equiv B) \Rightarrow$ T. Otherwise $(A \equiv B) \Rightarrow$ F.

Note: This rule just guarantees that $(A \equiv C)$ is truth-functionally equivalent to $((A \supset C) \wedge (C \supset A))$:

A	B	$(A \equiv B)$	$((A \supset C) \wedge (C \supset A))$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Also note: $(A \supset C)$ is truth-functionally equivalent to $(\neg A \vee C)$:

A	B	$(A \supset C)$	$(\neg A \vee C)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

So: "$\supset$" and "$\equiv$" are redundant connectives.

But: Adding them to **PL** simplifies many expressions.

So: Create a new language by adding "$\supset$" and "$\equiv$" to **PL**. Call it **PLC**.

The Language PLC

A. Alphabet (15 symbols)

(1)	P Q R S ′	simple propositions
(2)	$\wedge$ $\vee$ $\neg$ $\supset$ $\equiv$	connectives
(3)	( )	punctuation
(4)	, $\therefore$ *	additional punctuation

B. Grammar

Definition of *atomic wff*:

(A1) P, Q, R, S are atomic *wffs*.
(A2) Any atomic *wff* followed by a prime ′ is an atomic *wff*.
(A3) Nothing else is an atomic *wff*.

Definition of *wff*:

(W1) Any atomic *wff* is a *wff*.
(W2) If A is a *wff*, so is $\neg A$.
(W3) If A and B are *wffs*, so is $(A \wedge B)$.
(W4) If A and B are *wffs*, so is $(A \vee B)$.
(W5) If A and B are *wffs*, so is $(A \supset B)$.
(W6) If A and B are *wffs*, so is $(A \equiv B)$.
(W7) Nothing else is a *wff*.

Semantic Rules for connectives in PLC

(P1) For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ T, then $(A \wedge B) \Rightarrow$ T. Otherwise $(A \wedge B) \Rightarrow$ F.

(P2) For any *wffs* A, B, if $A \Rightarrow$ F and $B \Rightarrow$ F, then $(A \vee B) \Rightarrow$ F. Otherwise $(A \vee B) \Rightarrow$ T.

(P3) For any *wffs* A, B, if $A \Rightarrow$ T, then $\neg A \Rightarrow$ F. Otherwise $\neg A \Rightarrow$ T.

(P4) For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ F, then $(A \supset B) \Rightarrow$ F. Otherwise $(A \supset B) \Rightarrow$ T.

(P5) For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ T, or if $A \Rightarrow$ F and $B \Rightarrow$ F, then $(A \equiv B) \Rightarrow$ T. Otherwise $(A \equiv B) \Rightarrow$ F.

Important Distinctions

(1) $(\mathsf{P} \wedge \mathsf{Q}) \therefore \mathsf{Q}$ is an argument (string) in **PLC**.

(2) $((\mathsf{P} \wedge \mathsf{Q}) \supset \mathsf{Q})$ is a *wff* in **PLC**.

(3) $(\mathsf{P} \wedge \mathsf{Q}) \vDash \mathsf{Q}$ is English for "$(\mathsf{P} \wedge \mathsf{Q})$ tautologically entails Q".

Claims

(1) $A \vDash C$ if and only if $\vDash (A \supset C)$.

*"The argument in **PLC** $A \therefore C$ is tautologically valid if and only if the conditional $(A \supset C)$ is a tautology."*

Proof of (1).

I. "$\Rightarrow$" (the "only if" part of the claim)

Suppose: $A \vDash C$.

Then: There is no valuation of the atoms in A and C in which $A \Rightarrow$ T and $C \Rightarrow$ F.

So: There is no valuation in which $(A \supset C)$ is false.

So: $(A \supset C)$ is a tautology.

II. "$\Leftarrow$" (the "if" part of the claim)

Suppose: $(A \supset C)$ is a tautology.

Then: There is no valuation of the atoms in A and C in which $(A \supset C)$ is false.

So: There is no valuation in which A is true and C is false.

So: $A \vDash C$.

(2) $A_1, A_2, ..., A_n \vDash C$ if and only if $\vDash((A_1 \wedge A_2 \wedge ... \wedge A_n) \supset C)$.

Let $A_1, A_2, ..., A_n \therefore C$ be an argument in **PLC**. Then its ***corresponding conditional*** is the **PLC** *wff* $((A_1 \wedge A_2 \wedge ... \wedge A_n) \supset C)$.

Proof of (2).

I. "⇒"

Suppose: $A_1, A_2, ..., A_n \vDash C$.

Then: There is no valuation of the atoms in $A_1, A_2, ..., A_n, C$ in which $A_1, A_2, ..., A_n$ are all true and C is false.

So: There is no valuation that makes $(A_1 \wedge A_2 \wedge ... \wedge A_n)$ true and C false.

So: There is no valuation that makes $((A_1 \wedge A_2 \wedge ... \wedge A_n) \supset C)$ false.

So: $\vDash((A_1, A_2, ..., A_n) \supset C)$.

II. "⇐": *For homework!*