Chapters 12, 13: Tautologies and Tautological Entailment

A *wff* of **PL** is a ***tautology*** if it takes the value true on *every* valuation of its atoms.

A *wff* of **PL** is a ***contradiction*** if it takes the value false on *every* valuation of its atoms.

Examples of tautologies:

(1) $\neg(P \wedge \neg P)$

(2) $(P \vee \neg P)$

(3) $((P \wedge Q) \vee (\neg P \vee \neg Q))$

Check (3):

P	Q	((P	∧	Q)	∨	(¬P	∨	¬Q))
T	T		T		T		F	
T	F		F		T		T	
F	T		F		T		T	
F	F		F		T		T	

Claims:

(a) The negation of a tautology is a contradiction.

(b) The negation of a contradiction is a tautology.

(c) If the conclusion of an argument is a tautology, then the argument is valid.

Testing Arguments for Validity in PL

Ex1. (1) Jack is logical.

(2) It isn't the case that Jack is logical but Jill isn't.

(3) Jill is logical.

Translation key:

Let P = Jack is logical

Q = Jill is logical

(1) P

(2) $\neg(\mathsf{P} \wedge \neg\mathsf{Q})$

(3) Q

		(1)	(2)	(3)
P	Q	P	$\neg(\mathsf{P} \wedge \neg\mathsf{Q})$	Q
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

There's no row in which the premises are all T and the conclusion is F

So: Argument is VALID!

Ex2. (1) Either Jack or Jill went up the hill.

(2) It isn't the case that Jack went up thehill and Jo didn't.

(3) Hence it isn't the case that Jo went up the hill and Jill didn't.

Translation key:

Let P = Jack went up the hill

Q = Jill went up the hill

R = Jo went up the hill

(1) $(\mathsf{P} \vee \mathsf{Q})$

(2) $\neg(\mathsf{P} \wedge \neg\mathsf{R})$

(3) $\neg(\mathsf{R} \wedge \neg\mathsf{Q})$

			(1)	(2)	(3)
P	Q	R	$(\mathsf{P} \vee \mathsf{Q})$	$\neg(\mathsf{P} \wedge \neg\mathsf{R})$	$\neg(\mathsf{R} \wedge \neg\mathsf{Q})$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	F
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	F	T	T

premises are true and conclusion is false

For $\mathsf{P} \Rightarrow$ T, $\mathsf{Q} \Rightarrow$ F, $\mathsf{R} \Rightarrow$ T, the argument has all true premises and false conclusion. So it is INVALID!

The **PL** *wffs* A_1, A_2, ..., A_n ***tautologically entail*** the *wff* C just when there is no valuation of the atoms in A_1, A_2, ..., A_n and C which makes A_1, A_2, ..., A_n all true and C false.

Ex1: The **PL** *wffs* P, ¬(P ∧ ¬Q) tautologically entail the **PL** *wff* Q.

Ex2: The **PL** *wffs* (P ∨ Q), ¬(P ∧ ¬R) do not tautologically entail the **PL** *wff* ¬(R ∧ ¬Q).

Expressing Arguments in PL

- Use the symbol "," to separate premises in **PL**.
- Use the symbol "∴" to indicate a conclusion in **PL**.

Ex1: P, ¬(P ∧ ¬Q) ∴ Q

Ex2: (P ∨ Q), ¬(P ∧ ¬R) ∴ ¬(R ∧ ¬Q)

- These strings of symbols are completely in the language **PL**.
- Technically, we could add another Rule of Grammar that precisely defines such "argument strings". (These strings are not *wffs*!)

An argument (string) in **PL** is ***tautologically valid*** just when its premises tautologically entail its conclusion.

Ex3: $(P \vee Q), (R \vee \neg P), (\neg\neg R \vee \neg Q) \therefore R$

			(1)	(2)	(3)	(4)
P	Q	R	$(P \vee Q)$	$(R \vee \neg P)$	$(\neg\neg R \vee \neg Q)$	R
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	T	T	T	T
T	F	F	T	F	T	F
F	T	T	T	T	T	T
F	T	F	T	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

No row with all true premises and false conclusion.

So: Premises *tautologically entail* conclusion.

So: Argument in **PL** is *tautologically valid*.

Ex4: $(\neg P \vee R), (P \vee Q), \neg(Q \wedge \neg S) \therefore (R \vee S)$

				(1)	(2)	(3)	(4)
P	Q	R	S	$(\neg P \vee R)$	$(P \vee Q)$	$\neg(Q \wedge \neg S)$	$(R \vee S)$
T	T	T	T				
T	T	T	F				
T	T	F	T				
T	T	F	F	F	T		F
T	F	T	T				
T	F	T	F				
T	F	F	T				
T	F	F	F	F	T		F
F	T	T	T				
F	T	T	F				
F	T	F	T				
F	T	F	F	T	T	F	F
F	F	T	T				
F	F	T	F				
F	F	F	T				
F	F	F	F		F		F

1. *Evaluate conclusion.*
2. *Evaluate simplest premise.*
3. *Evaluate other premises.*

No row with all true premises and false conclusion.

So: Premises *tautologically entail* conclusion.

So: Argument in **PL** is *tautologically valid.*

Terminology:
Let "$\vDash$" be a symbol in English that is shorthand for "tautologically entails".

So: $A_1, A_2, \ldots, A_n \vDash C$ is shorthand in English for "The **PL** *wffs* $A_1, A_2, \ldots, A_n$ tautologically entail the **PL** *wff* C."

Important!

1. "$\vDash$" is not a symbol in **PL**.
2. "A_1", "A_2", "A_n", "C" are not symbols in **PL**.

These are symbols in English and are used to talk about ***PL****.*

So: $(\mathsf{P} \vee \mathsf{Q}), \neg\mathsf{P} \vDash \mathsf{Q}$ is shorthand English (and not **PL**) for "The **PL** *wffs* $(\mathsf{P} \vee \mathsf{Q}), \neg\mathsf{P}$ tautologically entail the **PL** *wff* Q."

But: $(\mathsf{P} \vee \mathsf{Q}), \neg\mathsf{P} \therefore \mathsf{Q}$ *is* an expression in **PL**. It's an argument string in **PL**.

Again: $\therefore$ is a symbol in **PL**. $\vDash$ is a symbol in English.

Terminology:
Let $\vDash C$ mean in English "The **PL** *wff* C is a tautology."

A set of **PL** *wffs* A_1, A_2, ..., A_n is ***satisfiable*** if there is a valuation of their atoms that makes them all true.

A set of **PL** *wffs* A_1, A_2, ..., A_n is ***tautologically inconsistent*** if it is not satisfiable.

There is no valuation of their atoms that makes them all true.

Some Claims:

(a) A_1, A_2, ..., $A_n \vDash C$ if and only if A_1, A_2, ..., A_n, $\neg C$ are *tautologically inconsistent*.

Proof: A_1, A_2, ..., $A_n \vDash C$ if and only if there's no valuation that makes A_1, A_2, ..., A_n true and C false.

Or: There's no valuation that makes A_1, A_2, ..., A_n true and $\neg C$ true.

(b) For any **PL** *wff* B, if $A_1, A_2, \ldots, A_n \vDash C$ then $A_1, A_2, \ldots, A_n, B \vDash C$.

- In other words, if the truth of $A_1, A_2, \ldots, A_n$ entails the truth of C, then the truth of $A_1, A_2, \ldots, A_n$ and B entails the truth of C.
- *Adding "extra info" to a valid argument doesn't affect its validity.*

Ex. (1) All animals with wings can fly.
(2) Pigs have wings.
(3) Pigs can fly.

Add: (2.5) Wilbur has wings.
Or: (2.5) Pigs don't have wings.
Or: (2.5) Pigs can't fly.

Proof:

Suppose: $A_1, A_2, \ldots, A_n \vDash C$ and B is any *wff*.

Then: *Whenever* $A_1, A_2, \ldots, A_n$ are all true, so is C.

Now: Consider *any* valuation that makes $A_1, A_2, \ldots, A_n$ true and B true.

Then: Under this valuation, C will also be true.

So: Any valuation that makes $A_1, A_2, \ldots, A_n$ true and B true will also make C true. In other words, $A_1, A_2, \ldots, A_n, B \vDash C$.

(c) If C is a tautology, then $A_1, A_2, \ldots, A_n \vDash C$, for *any* A_i.

A tautology is tautologically entailed by any set of premises.

(d) Any **PL** *wffs* A and B are truth-functionally equivalent just when $A \vDash B$ and $B \vDash A$.

(e) If C is a contradiction and $A_1, A_2, \ldots, A_n, B \vDash C$, then $A_1, A_2, \ldots, A_n \vDash \neg B$.

Proof:

Suppose: C is a contradiction and $A_1, A_2, \ldots, A_n, B \vDash C$.

Then: Whenever $A_1, A_2, \ldots, A_n, B$ are all true, so is C.

But: C is never true.

So: $A_1, A_2, \ldots, A_n, B$ can never all be true.

So: Whenever $A_1, A_2, \ldots, A_n$ are all true, B must be false.

Thus: Whenever $A_1, A_2, \ldots, A_n$ are all true, $\neg B$ must be true.

Thus: $A_1, A_2, \ldots, A_n \vDash \neg B$.