Chapter 11: Truth Functions

A way of forming a complex sentence out of one or more constituent sentences is ***truth-functional*** if fixing the truth-values of the constituent sentences is always enough to determine the truth-value of the complex sentence.

Claim: Every *wff* of **PL** is a truth-functional combination of atomic *wffs*.

Why?
- Tree constructions are unique.
- The Semantic Rules for the connectives guarantee that the truth-value of a conjunction, disjunction, or negation *only* depends on the truth-values of its atomic *wffs*.

Truth Tables and Valuations:

Task: Use a truth table to determine all possible truth-values of a given *wff*.

Ex. $\neg(P \wedge \neg(\neg Q \vee R))$

P	Q	R	$\neg(P \wedge \neg(\neg Q \vee R))$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

A ***valuation*** of a *wff* is an assignment of truth values to its atomic *wffs*.

So: Each row in a truth table for a *wff* is a valuation for it.

Truth-Functional Equivalence

> The **PL** *wffs* A and B are ***truth-functionally equivalent*** just if, on each valuation of all the atoms occuring in them, A and B take the same value.

Ex1. $\neg(\mathsf{P} \vee \mathsf{Q})$, $(\neg\mathsf{P} \wedge \neg\mathsf{Q})$, $\neg(\mathsf{P} \wedge \mathsf{Q})$, $(\neg\mathsf{P} \vee \neg\mathsf{Q})$

P	Q	$\neg(\mathsf{P} \vee \mathsf{Q})$	$(\neg\mathsf{P} \wedge \neg\mathsf{Q})$	$\neg(\mathsf{P} \wedge \mathsf{Q})$	$(\neg\mathsf{P} \vee \neg\mathsf{Q})$
T	T	F	F	F	F
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T

truth-functionally equivalent   *truth-functionally equivalent*

Ex2. $(\mathsf{P} \vee \mathsf{Q})$, $\neg(\neg\mathsf{P} \wedge \neg\mathsf{Q})$, $(\mathsf{P} \wedge \mathsf{Q})$, $\neg(\neg\mathsf{P} \vee \neg\mathsf{Q})$

P	Q	$(\mathsf{P} \vee \mathsf{Q})$	$\neg(\neg\mathsf{P} \wedge \neg\mathsf{Q})$	$(\mathsf{P} \wedge \mathsf{Q})$	$\neg(\neg\mathsf{P} \vee \neg\mathsf{Q})$
T	T	T	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	F	F

truth-functionally equivalent   *truth-functionally equivalent*

Expressive Equivalence

Claim: *Any* possible truth-functional combination of atomic *wffs* is truth-functionally equivalent to a *wff* in **PL** constructed using just the three connectives $\wedge$, $\vee$, $\neg$.

What this means: Any *truth-functional* way of combining atomic sentences to form compound sentences (using any sort of connectives you wish), is equivalent to using some combination of the three connectives, $\wedge$, $\vee$, $\neg$, of **PL**.

Terminology:

1. A ***basic conjunction*** of a set of atomic *wffs* is a *wff* formed by conjoining all the members of the set, or their negations, but not both.

   Ex. Basic conjunctions of $\{\mathsf{P}, \mathsf{Q}, \mathsf{S}, \mathsf{P}'\}$:

   (a) $(\mathsf{P} \wedge \neg\mathsf{Q} \wedge \neg\mathsf{S} \wedge \mathsf{P}')$

   (b) $(\neg\mathsf{P} \wedge \neg\mathsf{Q} \wedge \mathsf{S} \wedge \neg\mathsf{P}')$

   (c) $(\neg\mathsf{P} \wedge \neg\mathsf{Q} \wedge \neg\mathsf{S} \wedge \neg\mathsf{P}')$

   A basic conjunction is true on only one assignment of truth values to its atoms.

   (a) is true only when $\mathsf{P} \Rightarrow$ T, $\mathsf{Q} \Rightarrow$ F, $\mathsf{S} \Rightarrow$ F, $\mathsf{P}' \Rightarrow$ T.

   (b) is true only when $\mathsf{P} \Rightarrow$ F, $\mathsf{Q} \Rightarrow$ F, $\mathsf{S} \Rightarrow$ T, $\mathsf{P}' \Rightarrow$ F.

   (c) is true only when $\mathsf{P} \Rightarrow$ F, $\mathsf{Q} \Rightarrow$ F, $\mathsf{S} \Rightarrow$ F, $\mathsf{P}' \Rightarrow$ F.

2. A ***truth-function*** is a function that takes the truth-values of atomic *wff* as input and outputs a truth-value.

So: Truth-functions correspond to truth-tables!

Proof of Claim

Task: Show that for *any* given truth function, represented by *any arbitrary* truth table, we can write down a **PL** *wff* with exactly that truth table.

Three Possible Cases:

Case 1: The truth-function has all F's in its truth table.

How to construct the corresponding **PL** *wff*:

(1) Take each atomic *wff* in the truth table and conjoin it with its negation.

(2) Form the disjunction of all of the conjunctions in Step (1).

Ex: Suppose the truth function !(P, Q, R, S) is given by the truth table:

P	Q	R	S	!(P, Q, R, S)
				F
				F
				F
				⋮

Then: The **PL** *wff* truth-functionally equivalent to it is:

$((P \wedge \neg P) \vee (Q \wedge \neg Q) \vee (R \wedge \neg R) \vee (S \wedge \neg S))$

Case 2: The truth-function has exactly one T in its truth table.

How to construct the corresponding **PL** *wff*:
(1) Construct the *basic conjunction* corresponding to the valuation of the truth-function that makes it true.

Ex: Suppose the truth function %(P, Q, R, S) is given by the truth table:

P	Q	R	S	%(P, Q, R, S)
				F
F	T	F	T	T
				F
				⋮

Then: The **PL** *wff* truth-functionally equivalent to it is:

$(\neg P \wedge Q \wedge \neg R \wedge S)$

Case 3: The truth-function has T in more than one row in its truth table.

How to construct the corresponding **PL** *wff*:

(1) Construct each basic conjunction that corresponds to each valuation that makes the truth-function true.

(2) Form the disjunction of all the basic conjunctions in Step (1).

Ex:

P	Q	R	$(P, Q, R, S)	*basic conjunctions*
T	T	T	F	
T	T	F	T	$(P \wedge Q \wedge \neg R)$
T	F	T	F	
T	F	F	F	
F	T	T	T	$(\neg P \wedge Q \wedge R)$
F	T	F	T	$(\neg P \wedge Q \wedge \neg R)$
F	F	T	T	$(\neg P \wedge \neg Q \wedge R)$
F	F	F	T	$(\neg P \wedge \neg Q \wedge \neg R)$

Then: The **PL** *wff* truth-functionally equivalent to $(P, Q, R, S) is

$$((P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R))$$

A set of connectives is ***expressively adequate*** if a language containing just those connectives is rich enough to express all truth-functions of the atomic *wffs* of the language.

So: The standard set of **PL** connectives $\{\wedge, \vee, \neg\}$ is expressively adequate.

Claim: The following sets of connectives are expressively adequate:
(a) $\{\wedge, \neg\}$
(b) $\{\vee, \neg\}$

Proof: For (a), recall that any *wff* of the general form $(A \vee B)$ is truth-functionally equivalent to $\neg(\neg A \wedge \neg B)$.
For (b), recall that any *wff* of the general form $(A \wedge B)$ is truth-functionally equivalent to $\neg(\neg A \vee \neg B)$.

Claim: The set $\{\wedge, \vee\}$ is not expressively adequate.

Proof:

A	B	$(A \wedge B)$	$(A \vee B)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Can't replicate negation!

- *Conjunctions and disjunctions are always true when their atoms are true!*
- *But*: $\neg A$ *is false when atom is true.*

The "nand" connective. $\downarrow$

Semantic Rule for $\downarrow$

For any *wffs* A, B, if $A \Rightarrow$ F and $B \Rightarrow$ F, then $(A \downarrow B) \Rightarrow$ T. Otherwise $(A \downarrow B) \Rightarrow$ F.

A	B	$(A \downarrow B)$
T	T	F
T	F	F
F	T	F
F	F	T

"A nand B" means "Neither A nor B" (or $\neg(\mathsf{P} \vee \mathsf{Q})$).

Claims:

(1) $\neg A$ is truth-functionally equivalent to $(A \downarrow A)$.

A	$\neg A$	$(A \downarrow A)$
T	F	F
F	T	T

(2) $(A \vee B)$ is truth-functionally equivalent to $((A \downarrow A) \downarrow (B \downarrow B))$.

A	B	$(A \wedge B)$	$((A \downarrow A) \downarrow (B \downarrow B))$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

So: Since $\{\wedge, \neg\}$ is expressively adequate, so is $\{\downarrow\}$!

The "Sheffer stroke" connective. |

Semantic Rule for |

For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ T, then $(A \mid B) \Rightarrow$ F. Otherwise $(A \mid B) \Rightarrow$ T.

A	B	$(A \mid B)$
T	T	F
T	F	T
F	T	T
F	F	T

"A stroke B" means "Not both A and B" (or $\neg(\mathsf{P} \wedge \mathsf{Q})$).

Claims:

(1) $\neg A$ is truth-functionally equivalent to $(A \mid A)$.

A	$\neg A$	$(A \mid A)$
T	F	F
F	T	T

(2) $(A \vee B)$ is truth-functionally equivalent to $((A \mid A) \mid (B \mid B))$.

A	B	$(A \vee B)$	$((A \mid A) \mid (B \mid B))$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

So: Since $\{\vee, \neg\}$ is expressively adequate, so is $\{|\}$!