Chapter 8: The Syntax of PL

I. Syntactic Rules for PL

A. Alphabet (13 symbols)

(1)	P Q R S ′	simple propositions
(2)	$\wedge$ $\vee$ $\neg$	connectives
(3)	( )	punctuation
(4)	, $\therefore$ *	additional punctuation

B. Grammar: What counts as a "well-formed formula" (wff).

Definition of an "atomic well-formed formula" (atomic *wff*):

> (A1) P, Q, R, S are atomic *wffs*.
>
> (A2) Any atomic *wff* followed by a prime ′ is an atomic *wff*.
>
> (A3) Nothing else is an atomic *wff*.

Exs:

P

P′

P′′′′′

Q′′′

Definition of a "well-formed formula" (*wff*):

(W1)	Any atomic *wff* is a *wff*.
(W2)	If A is a *wff*, so is $\neg A$.
(W3)	If A and B are *wffs*, so is $(A \wedge B)$.
(W4)	If A and B are *wffs*, so is $(A \vee B)$.
(W5)	Nothing else is a *wff*.

II. Construction Trees

Ex1. $\neg(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R}))$ ***Wff? Yes!***

Informal proof that this is a wff:

(1)	Q is a *wff*.	(A1), (W1)
(2)	$\neg\mathsf{Q}$ is a *wff*.	(W2), (1)
(3)	R is a *wff*.	(A1), (W1)
(4)	$(\neg\mathsf{Q} \vee \mathsf{R})$ is a *wff*.	(W4), (2), (3)
(5)	$\neg(\neg\mathsf{Q} \vee \mathsf{R})$ is a *wff*.	(W2), (4)
(6)	P is a *wff*.	(A1), (W1)
(7)	$(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R}))$ is a *wff*.	(W3), (5), (6)
(8)	$\neg(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R}))$ is a *wff*.	(W2), (7)

Construction tree method:

Q

$\neg Q \qquad R$

$(\neg Q \vee R)$

$P \qquad \neg(\neg Q \vee R)$

$(P \wedge \neg(\neg Q \vee R))$

$\neg(P \wedge \neg(\neg Q \vee R))$

If branch tips are simple declarative sentences, then expression is a wff!

Trunk is the expression under consideration.

Ex2. $((P \vee (R \wedge \neg S)) \vee \neg(Q \wedge \neg P))$

$S \qquad\qquad P$

$R \quad \neg S \qquad Q \quad \neg P$

$P \quad (R \wedge \neg S) \qquad (Q \wedge \neg P)$

$(P \vee (R \wedge \neg S)) \qquad \neg(Q \wedge \neg P))$

$((P \vee (R \wedge \neg S)) \vee \neg(Q \wedge \neg P))$

III. Main Connectives

Def. 1. The ***main connective*** of a non-atomic *wff* is the connective introduced at the last stage of its construction tree.

IV. Subformula and Scope

Def. 2. A *wff* S is a ***subformula*** of a *wff* A if S appears anywhere on the construction tree for A.

Def. 3. The ***scope*** of a connective is the *wff* on the construction tree where the connective is introduced.

Chapter 9: The Semantics of PL

Idea: Valid-deductive arguments are those in which, if the premises are true, the conclusion must be true.

So: Need a way of assigning truth-values to *wffs* in PL.

Terminology: Abbreviate "A is true" by "$A \Rightarrow$ T"

Abbreviate "A is false" by "$A \Rightarrow$ F"

Semantic Rules for Connectives

(P1) For any *wffs* A, B, if $A \Rightarrow$ T and $B \Rightarrow$ T, then $(A \wedge B) \Rightarrow$ T. Otherwise $(A \wedge B) \Rightarrow$ F.

(P2) For any *wffs* A, B, if $A \Rightarrow$ F and $B \Rightarrow$ F, then $(A \vee B) \Rightarrow$ F. Otherwise $(A \vee B) \Rightarrow$ T.

(P3) For any *wffs* A, B, if $A \Rightarrow$ T, then $\neg A \Rightarrow$ F. Otherwise $\neg A \Rightarrow$ T.

Truth Table Representation of Rules:

A	B	$(A \wedge B)$	$(A \vee B)$	$\neg$A
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Calculating Truth Values

Ex1: $\neg(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R}))$

suppose: $\mathsf{P} \Rightarrow \mathrm{T}$, $\mathsf{Q} \Rightarrow \mathrm{F}$, $\mathsf{R} \Rightarrow \mathrm{T}$

(1)	$\mathsf{Q} \Rightarrow \mathrm{F}$.	premise
(2)	$\neg\mathsf{Q} \Rightarrow \mathrm{T}$.	(P3), (1)
(3)	$\mathsf{R} \Rightarrow \mathrm{T}$.	premise
(4)	$(\neg\mathsf{Q} \vee \mathsf{R}) \Rightarrow \mathrm{T}$.	(P2), (2), (3)
(5)	$\neg(\neg\mathsf{Q} \vee \mathsf{R}) \Rightarrow \mathrm{F}$.	(P3), (4)
(6)	$\mathsf{P} \Rightarrow \mathrm{T}$.	premise
(7)	$(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R})) \Rightarrow \mathrm{F}$.	(P1), (5), (6)
(8)	$\neg(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R})) \Rightarrow \mathrm{T}$.	(P3), (7)

Informal Proof Hint:
Work from inner-most parentheses out.

$$\dfrac{\dfrac{\mathsf{P} \Rightarrow \mathrm{T} \qquad \dfrac{\dfrac{\dfrac{\mathsf{Q} \Rightarrow \mathrm{F}}{\neg\mathsf{Q} \Rightarrow \mathrm{T}} \qquad \mathsf{R} \Rightarrow \mathrm{T}}{(\neg\mathsf{Q} \vee \mathsf{R}) \Rightarrow \mathrm{T}}}{\neg(\neg\mathsf{Q} \vee \mathsf{R}) \Rightarrow \mathrm{F}}}{(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R})) \Rightarrow \mathrm{F}}}{\neg(\mathsf{P} \wedge \neg(\neg\mathsf{Q} \vee \mathsf{R})) \Rightarrow \mathrm{T}}$$

Tree form Hint:
Introduce one branch tip for every atomic wff.

Short-Cut: $(\mathsf{P} \Rightarrow \mathrm{T},\ \mathsf{Q} \Rightarrow \mathrm{F},\ \mathsf{R} \Rightarrow \mathrm{T})$

P	Q	R	¬	(P	∧	¬	(¬	Q	∨	R))
T	F	T	**T**	T	F	F	T	F	T	T
			8	6	7	5	2	1	4	3

Ex2: $((\mathsf{P} \vee (\mathsf{R} \wedge \neg \mathsf{S})) \wedge \neg(\mathsf{Q} \wedge \neg \mathsf{P}))$

suppose: $\mathsf{P} \Rightarrow \mathrm{F},\ \mathsf{Q} \Rightarrow \mathrm{F},\ \mathsf{R} \Rightarrow \mathrm{T},\ \mathsf{S} \Rightarrow \mathrm{F}$

$$\dfrac{\dfrac{\mathsf{P} \Rightarrow \mathrm{F} \qquad \dfrac{\mathsf{R} \Rightarrow \mathrm{T} \qquad \dfrac{\mathsf{S} \Rightarrow \mathrm{F}}{\neg \mathsf{S} \Rightarrow \mathrm{T}}}{(\mathsf{R} \wedge \neg \mathsf{S}) \Rightarrow \mathrm{T}}}{(\mathsf{P} \vee (\mathsf{R} \wedge \neg \mathsf{S})) \Rightarrow \mathrm{T}} \qquad \dfrac{\dfrac{\mathsf{Q} \Rightarrow \mathrm{F} \qquad \dfrac{\mathsf{P} \Rightarrow \mathrm{F}}{\neg \mathsf{P} \Rightarrow \mathrm{T}}}{(\mathsf{Q} \wedge \neg \mathsf{P}) \Rightarrow \mathrm{F}}}{\neg(\mathsf{Q} \wedge \neg \mathsf{P}) \Rightarrow \mathrm{T}}}{((\mathsf{P} \vee (\mathsf{R} \wedge \neg \mathsf{S})) \wedge \neg(\mathsf{Q} \wedge \neg \mathsf{P})) \Rightarrow \mathrm{T}}$$

Ex3: $\neg(P \vee ((Q \wedge \neg P) \vee R))$

Construction tree:

$$\dfrac{\dfrac{P \qquad \dfrac{\dfrac{Q \qquad \dfrac{P}{\neg P}}{(Q \wedge \neg P)} \qquad R}{(Q \wedge \neg P) \vee R}}{(P \vee ((Q \wedge \neg P) \vee R))}}{\neg(P \vee ((Q \wedge \neg P) \vee R))}$$

suppose: $P \Rightarrow T,\ R \Rightarrow T,\ Q \Rightarrow F$

$$\dfrac{\dfrac{P \Rightarrow T \qquad \dfrac{\dfrac{Q \Rightarrow F \qquad \dfrac{P \Rightarrow T}{\neg P \Rightarrow F}}{(Q \wedge \neg P) \Rightarrow F} \qquad R \Rightarrow T}{(Q \wedge \neg P) \vee R) \Rightarrow T}}{(P \vee ((Q \wedge \neg P) \vee R)) \Rightarrow T}}{\neg(P \vee ((Q \wedge \neg P) \vee R)) \Rightarrow F}$$

Short-Cut:

P	Q	R	¬	(	P	∨	(	(	Q	∧	¬	P	)	∨	R	)	)
T	F	T	**F**		T	T			F	F	F	T		T	T		
			9		7	8			3	4	2	1		6	5		