13. Maxwell's Equations and EM Waves.

Hunt (1991), Chaps 5 & 6

A. The Energy of an Electromagnetic Field.

- *1880s revision of Maxwell*: Guiding principle = concept of energy flow.
- Evidence for energy flow through seemingly empty space:
  - *induced currents*
  - *air-core transformers, condensers.*
- *But*: *Where* is this energy located?

Two equivalent expressions for electromagnetic energy of steady current:

$\frac{1}{2}\mathbf{A}\cdot\mathbf{J}$

- $\mathbf{A}$ = *vector potential*, $\mathbf{J}$ = *current density.*
- Suggests energy located in conductor.

$\frac{1}{2}\mu\mathbf{H}^2$

- μ = *permeability*, $\mathbf{H}$ = *magnetic force.*
- Suggests energy located outside conductor in magnetic field.

13. Maxwell's Equations and EM Waves.

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  - *induced currents*
  - *air-core transformers, condensers.*
- *But*: *Where* is this energy located?

Two equivalent expressions for electrostatic energy:

$\frac{1}{2}q\psi$

- q = *charge*, ψ = *electric potential.*
- Suggests energy located in charged object.

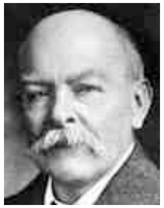
$\frac{1}{2}\varepsilon\mathbf{E}^2$

- ε = *permittivity*, $\mathbf{E}$ = *electric force.*
- Suggests energy located outside charged object in electric field.

- *Maxwell*: Treated potentials $\mathbf{A}$, ψ as fundamental quantities.

Poynting's account of energy flux

- Research project (1884):

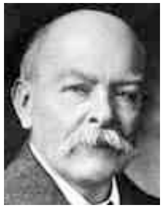


John Poynting
(1852-1914)

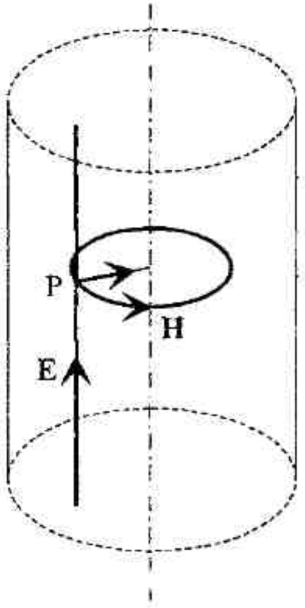
"How does the energy about an electric current pass from point to point -- that is, by what paths and acording to what law does it travel from the part of the circuit where it is first recognizable as electric and magnetic, to the parts where it is changed into heat and other forms?"

- *Solution*: The energy flux at each point in space is encoded in a vector **S** given by $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.

"...there is a general law for the transfer of energy, according to which it moves at any point perpendicularly to the plane containing the lines of electric force and magnetic force, and that the amount crossing unit of area per second of this plane is equal to the product of the intensities of the two forces, multiplied by the sine of the angle between them, divided by 4π; while the direction of flow of energy is that in which a right-handed screw would move if turned round from the positive direction of the electromotive to the positive direction of the magnetic intensity."

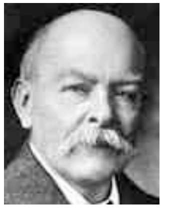


Ex. #1: Wire carrying a steady current

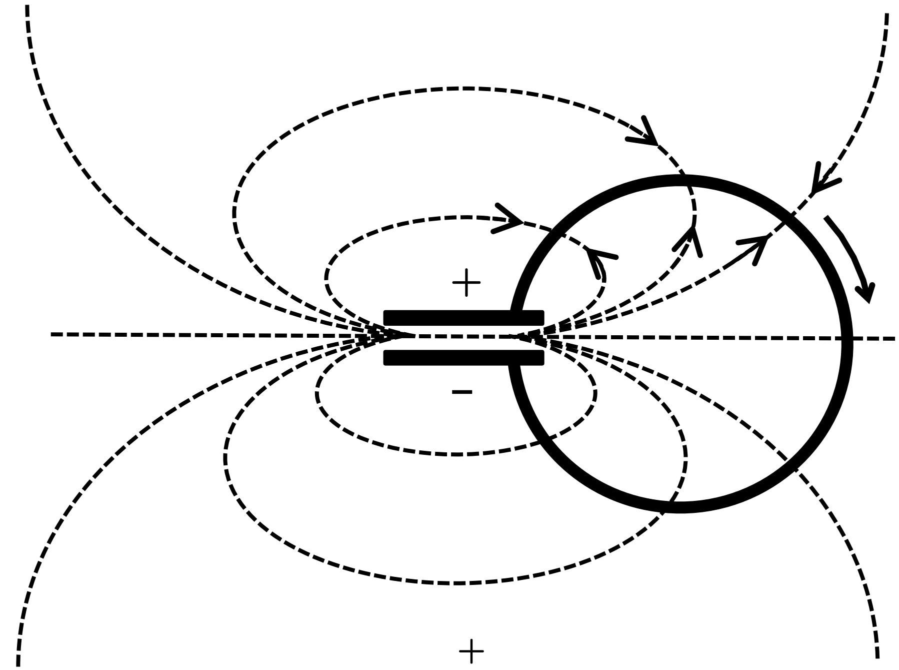


- **E** points in direction of current.
- **H** points in directions tangent to circle about wire.
- *So*: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ points radially towards wire.

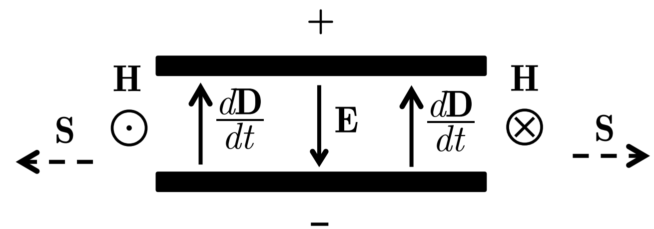
"I think it is necessary that we should realise thoroughly that if we accept Maxwell's theory of energy residing in the medium, we must no longer consider a current as something conveying energy along the con-ductor. A current in a conductor is rather to be regarded as consisting essentially of a convergence of electric and magnetic energy from the medium upon the conductor and its transformation there into other forms."



Ex. #2: Discharging condenser

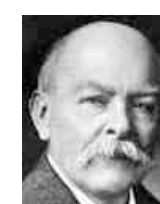


- Condenser discharged through high-resistance wire.
- Conduction current flows in wire from positive to negative plate.
- Electric field between plates decreases, producing displacement current $d\mathbf{D}/dt$ from negative to positive plate.
- *So*: Magnetic force **H** (determined by $d\mathbf{D}/dt$) points *into* plane on right side of condenser, and *out* of plane on left side.
- *So*: **S** points *out* from between plates.
- *And*: **S** points *in* around wire (**E** points in direction of current in wire).



- *Significance: Wire acts as an energy sink, and not an energy conduit.*

> "According to Maxwell's theory, currents consist essentially in a certain distribution of energy in and around a conductor, accompanied by the transformation and consequent movement of energy through the field."



B. Potentials and action-at-a-distance.

- *Recall*: Maxwell prioritizes vector potential $\mathbf{A}$.

"[**A**]... is the fundamental quantity in the theory of electromagnetism."

- *Let*: $J = \nabla \cdot \mathbf{A}$.
- *Maxwell shows*: $\dfrac{d^2 J}{dt^2} + \dfrac{d(\nabla^2 \psi)}{dt} = 0$.
- *...and claims*:

"$\nabla^2\psi$ which is proportional to the volume-density of the free electricity, is independent of t."

Which means: "The electric potential $[\psi]$ is determined solely by the spatial distribution of charge, which in a nonconductor does not change. This was the assumption usually made in electrostatics, and Maxwell simply extended it to general electromagnetic theory without alteration or explanation." (Hunt, pg. 117.)

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Implication #1: ψ changes *instantaneously* (independently of time) to changes in charge density.

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Implication #2: $d^2 J/dt^2 = 0$.

"J must be a linear function of t, or a constant, or zero, and we may therefore leave J and y out of account in considering wave disturbances."

- *Maxwell's typical prescription*: $J = \nabla \cdot \mathbf{A} = 0$.

- *Note*: The quantity $J = \nabla \cdot \mathbf{A}$ doesn't contribute to observable effects of Maxwell's theory.
    - *Which means: Any two descriptions that assign J different values (different "gauges") make all the same observable predictions.*

- *Maxwell's prescription*: $J = \nabla \cdot \mathbf{A} = 0$.
    - *"Coulomb gauge". Choice that makes electrostatic problems easy to handle, but makes propagation problems difficult.*

- *But*: No instantaneous propagation in FitzGerald's rubber-band model:

"Still there remains that the wheels seem to give a correct representation of the ether and that there can be no instantaneous propagation of anything in them."

- *FitzGerald's prescription (1888)*: $J = \nabla \cdot \mathbf{A} = -d\psi/dt$.
    - *"Lorentz gauge". Choice that makes propagation problems easy to handle.*

- *Entails*:

Homogeneous wave equations with finite propagation speeds!

$$\nabla^2\psi - \frac{d^2\psi}{dt^2} = 0, \qquad \nabla^2\mathbf{A} - \frac{d^2\mathbf{A}}{dt^2} = 0.$$

C. Heaviside's equations.

1884. Heaviside reformulates Maxwell's equations.

- Combine (2), (11): $\nabla \cdot \mu\mathbf{H} = 0$.
- Combine (10), (7): $\nabla \cdot \varepsilon\mathbf{E} = \rho$.
- *First circuital equation (Ampère's Law)*:
  - Combine (6), (8), (9): $\nabla \times \mathbf{H} = k\mathbf{E} + \varepsilon d\mathbf{E}/dt$.
- *Second circuital equation (Faraday's Law)*:
  - Restrict (3) to $\mathbf{v} = 0$: $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\psi$.
  - Take curl: $\nabla \times \mathbf{E} = -\nabla \times (\partial\mathbf{A}/\partial t) - \nabla \times (\boldsymbol{\nabla}\psi)$.
  - *Or*: $-\nabla \times \mathbf{E} = \partial/\partial t(\nabla \times \mathbf{A}) = \mu\partial\mathbf{H}/\partial t$.

"Maxwell's equations" (1873)

1. $\mathbf{B} = \nabla \times \mathbf{A}$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\mathbf{E} = -\partial\mathbf{A}/\partial t + \mathbf{v}\times(\nabla\times\mathbf{A}) - \nabla\psi$
4. $\mathbf{f} = \mathbf{J} \times \mathbf{B} - \rho\nabla\psi - m\nabla\Psi$
5. $\mathbf{B} = \mathbf{H} + \mathbf{M}$
6. $\nabla \times \mathbf{H} = \mathbf{J}$
7. $\mathbf{D} = \varepsilon\mathbf{E}$
8. $\mathbf{j} = k\mathbf{E}$
9. $\mathbf{J} = \mathbf{j} + \partial\mathbf{D}/\partial t$
10. $\nabla \cdot \mathbf{D} = \rho$
11. $\mathbf{B} = \mu\mathbf{H}$
12. $\nabla \cdot \mathbf{M} = m$
13. $\mathbf{H} = -\nabla\Psi$

Free space equations

$\nabla \cdot \varepsilon\mathbf{E} = 0 \qquad \nabla \times \mathbf{H} = \varepsilon\partial\mathbf{E}/\partial t$

$\nabla \cdot \mu\mathbf{H} = 0 \qquad -\nabla \times \mathbf{E} = \mu\partial\mathbf{H}/\partial t$

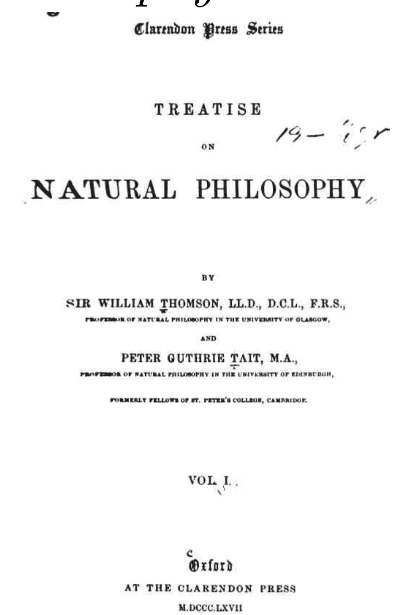
Equations for charges/currents:

$\nabla \cdot \varepsilon\mathbf{E} = \rho \qquad \nabla \times \mathbf{H} = k\mathbf{E} + \varepsilon\partial\mathbf{E}/\partial t$

$\nabla \cdot \mu\mathbf{H} = \sigma \qquad -\nabla \times \mathbf{E} = g\mathbf{H} + \mu\partial\mathbf{H}/\partial t$

- No evidence for magnetic charges σ or magnetic conduction currents $g\mathbf{H}$, but imposing symmetry on equations is analytically useful.
- $\mathbf{A}$ and ψ have been "murdered".

- *Motivation:* Thomson and Tait's (1867) *Treatise on Natural Philosophy*.


Clarendon Press Series

TREATISE

ON

NATURAL PHILOSOPHY

BY

SIR WILLIAM THOMSON, LL.D., D.C.L., F.R.S.,

PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW,

AND

PETER GUTHRIE TAIT, M.A.,

PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF EDINBURGH,

FORMERLY FELLOWS OF ST. PETER'S COLLEGE, CAMBRIDGE.

VOL. I.

Oxford

AT THE CLARENDON PRESS

M.DCCC.LXVII

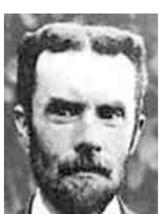
> *T & T' Principle of Activity*: The rate at which a force adds energy to a system is the product of the applied force and the velocity of the point at which it acts.

  - *In other words*: A force $\mathbf{F}$ acting on a mass m moving with velocity $\mathbf{v}$ will add energy at the rate $\mathbf{F} \cdot \mathbf{v} = m(d\mathbf{v}/dt) \cdot \mathbf{v}$, which is the rate dT/dt at which the kinetic energy $T = ½m\mathbf{v}^2$ increases.

- *Problem with Maxwell's theory*:

> "If we take Maxwell's equations and endeavour to immediately form the equation of activity... it will be found to be impossible. They will not work in the manner proposed."

- *Task*:



> "... to remodel Maxwell's equations in some important particulars... This done, the equation of activity is at once derivable from the two cross-connections of electric force and magnetic current, magnetic force and elecric current, in a manner analogous to $\mathbf{F} \cdot \mathbf{v} = dT/dt$, without roundabout work."

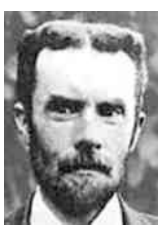
- *In Maxwell's theory, analog of $dT/dt = \mathbf{F} \cdot \mathbf{v}$ is:*

$$\frac{dW}{dt} = \mathbf{E} \cdot \varepsilon \frac{d\mathbf{E}}{dt} + \mathbf{H} \cdot \mu \frac{d\mathbf{H}}{dt}$$

- *In new formulation (free space)*: $\nabla \times \mathbf{H} = \varepsilon d\mathbf{E}/dt$, and $-\nabla \times \mathbf{E} = \mu \partial \mathbf{H}/\partial t$.

- *So*: $\dfrac{dW}{dt} = \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\nabla \cdot (\mathbf{E} \times \mathbf{H})$

- *And*: This reproduces Poynting's result: The rate of increase of the energy density W is equal to the convergence of the Poynting vector $\mathbf{E} \times \mathbf{H}$.



"[This is]... an entirely new way of developing Maxwell's electromagnetic scheme, with a new fundamental equation, $-\nabla \times \mathbf{E} = \mu \partial \mathbf{H}/\partial t$, abolishing ψ and $\mathbf{A}$ altogether, and working with $\mathbf{E}$ and $\mathbf{H}$ from the beginning."

D. Application to propagation of electric signals along wires.

- *Before mid-1880s*: Common view describes signals flowing within a wire in analogy with water in a pipe

  - *Thomson's "KR Law"*: The rate of signaling is limited solely by the product of the total capacitance (K) and resistance (R) of the line.
  - *1870s. Heaviside's Telegraph Equation*: Modifies Thomson's law to take account not only of resistance (R) and capacitance (S), but also inductance (L) and leakage conductance (K).

- *1880s*: Heaviside turns to his new formulation of Maxwell's theory.
- *Essence of Maxwellian approach*:

"According to Maxwell's view, there is a great deal more going on outside the conductor than inside it."

"[Maxwell's view]... will assist in abolishing the time-honoured but (in my opinion) essentially vicious practice of associating the electric current in a wire with the motion through the wire of a hypothetical *quasi*-substance, which is a pure invention that may well be dispensed with."

Maxwellian view of signal propagation (Hunt, pg. 132):

- "In this view, the main function of a conducting wire is to enable part of the accumulating electrical strain in the surrounding field to break down, thus preventing the electromagnetic machinery in the ether from jamming itself as it does in a condenser as it becomes fully charged."
- "By providing a dumping ground for part of the strain of the field, the wire keeps the lines of magnetic force wrapped tightly around itself, and serves to guide the main flow of energy from one place to another."

Problem of signal distortion.

- For EM waves, Mawell showed:
  - If electrical energy = magnetic energy, then no distortion.
  - *Conductive medium*: electrical energy dissipates as heat; magnetic energy dominates; thus distortion.

Equations for charges/currents:

$$\nabla \cdot \varepsilon \mathbf{E} = \rho \qquad \nabla \times \mathbf{H} = k\mathbf{E} + \varepsilon\, d\mathbf{E}/dt$$

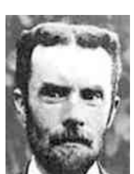
$$\nabla \cdot \mu \mathbf{H} = \sigma \qquad -\nabla \times \mathbf{E} = g\mathbf{H} + \mu \partial \mathbf{H}/\partial t$$

- *Now*: Heaviside's reformulation includes term for magnetic conduction current...
  - "...if the magnetic and electrical conductivities were adjusted so that each extracted energy from the wave at exactly the same rate, the electrical and magnetic energies of the wave would remain in balance, and the wave would move forward without distortion." (Hunt, pg. xx.)

- *But*:

"When I introduced the new property of matter [magnetic conductivity], it was merely to complete the analogy between the electric and magnetic sides of electro-magnetism. The property is non-existent, so far as I know."

- *However*:

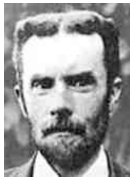


"I have recently found out how to precisely imitate its effect..."

How to mimick magnetic conductivity

- "When an electromagnetic wave runs along a wire, its moving magnetic field induces conduction currents in the wire; as these are dissipated by the electrical resistance of the wire, magnetic energy is extracted from the wave much as it would be by magnetic conductivity in the dielectric... If the loss of energy in the wire were made to occur at exactly the same rate as that in the surrounding dielectric owing to leakage conductance (as it could be, by making L/R, the time constant for the decay of the magnetic energy, equal to S/K, the time constant for the decay of the electric energy), then just as in the imaginary case of magnetic conductivity, the electrical and magnetic energies of the wave would remain in balance, and the wave, though attenuated, would propagate without distortion." (Hunt, pg. 134.)

"The relation $R/L = K/S$, which does not require excessive leakage when the wires are of copper of low resistance, removes the distortion otherwise suffered by the waves."

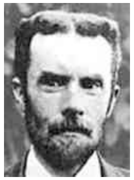


Telegraph Equation (contemporary view)

$$\frac{\partial^2 v}{\partial^2 t} + \left(\frac{R}{L} + \frac{K}{S}\right)\frac{\partial v}{\partial t} + \frac{RK}{LS} v = \frac{1}{(LS)^2}\frac{\partial^2 v}{\partial x^2}$$

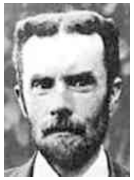
$R/L = K/S$ implies different frequencies that a signal (wave packet) consists of travel at same velocity and attenuate at same rate.

- *Now*: Realist telegraph and telephone lines are charaterized by $L/R \ll S/K$.
  - *Submarine cables*: High capacitance S due to iron sheathing and surrounding sea; low leakage conductance K due to gutta-percha insulation.
  - *Overhead lines*: Lower S and higher K, but still experience distortion at long distances.
- *Suggests to Heaviside*: Increase L to decrease distortion.



""Are there really any hopes for Atlantic telegraphy? Without any desire to be over sanguine, I think we may expect great advances in the future. Thus, without reducing the resistance or reducing the permittance (obvious ways of increasing speed), increase the leakage [K] as far as is consistent with other things, and increase the inductance [L] greatly."

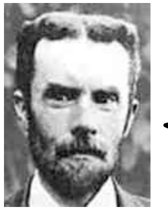
"A remarkable misconception on this point seems to be somewhat generally held. It seems to be imagnied that self-induction is harmful to long-distance telephony. The precise contrary is the case. It is the very life and soul of it, as is proved both by practical experience in America and on the Continent on very long copper circuits, and by examining the theory of matter. I have proved this in considerable detail; but they will not believe it. So far does the misconception extend that it has perhaps contributed to leading Mr. W. H. Preece to conclude that the coefficient of self-induction in copper circuits is negligible (several hundred times smaller than it can possibly be), on the basis of his recent remarkable experimental researches."



The Bouncer, Mr. Prigs, and Taffy

William Henry Preece (1834-1913)

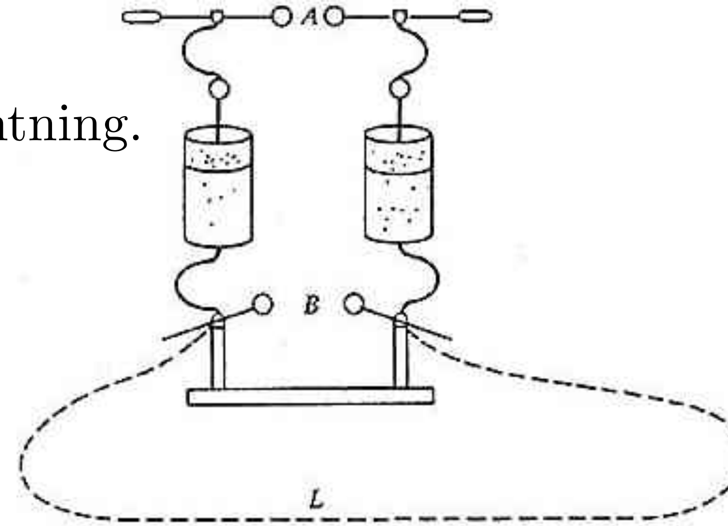
- *1887*. Heaviside and brother Arthur (Post Office electrician) attempt to publish a paper on inductive loading in *Journal for the Society of Telegraph-Engineers and Electricians*.
- Chief Post Office Engineer W. H. Preece supresses it!
  - *Endorses Thomson's KR law.*
  - *Ulterior motive: Campaign to replace iron telegraph/telephone lines with expensive, low inductance copper.*
  - *Recent debate between Preece and London engineering professors S. Thompson and W. E. Ayrton, and Preece over Preece's use of the KR law.*
- *Sept. 1887 meeting of the BAAS*: Preece presents paper on copper wires and importance of low inductance.



"dreadful stuff, rank quackery..."

  - *Heaviside against Preece: The reason why copper is reliable isn't due to its low inductance, but it's high ratio of inductance to resistance L/R.*
- *Oct. 1887*: Editor of *Electrician* is replaced. New editor discontinues Heaviside's series on Maxwell's theory (running since 1885).

Lodge and lightning to the rescue



- Low-reistance copper wire L.
- High-resistance air gap B.

- *1888*. Lodge lectures on experiments with lightning.
  - *"He found that when a [Leyden] jar was given two or more paths along which to discharge, a sudden flash did not simply take the 'easy' path offered by a thick copper rod; it rushed simultaneously along all available paths, even if that meant running along a high-resistance iron wire or sparking across an air gap." (Hunt, pg. 146.)*
- *Now*: Thomson (1850s) had shown:
  - *Discharge of a condenser (like a Leyden jar) oscillates with frequency determined by capacitance, inductance, and resistance of the discharging circuit.*
- *And*: D. E. Hughes (1886) had shown:
  - *A rapidly varying current is confined to outer surface of conductor ("skin effect"), and is subject to higher resistance and lower self-induction than a steady current.*
  - *Effect increases with increasing frequency of oscillation, and is greater in "magnetic metals" like iron than in "non-magnetic metals" like copper.*
- *Lodge thus claims*: Lightning discharges are oscillatory and pass through all available paths because of skin-effect in lightning rod conductors.
  - *And suggests:* *Cheap thin iron rod is just as effective as expensive thick copper rod.*

- Lodge adopts Heaviside's explanation of the skin-effect:

  - *Electromagnetic force and energy enter a conductor from surrounding field.*
  - *Thus*: Flow of current begins at boundary and works its way inward.
  - *And*: With rapid oscillations, self-induction prevents current from penetrating beyond outer skin before oscillation reverses.

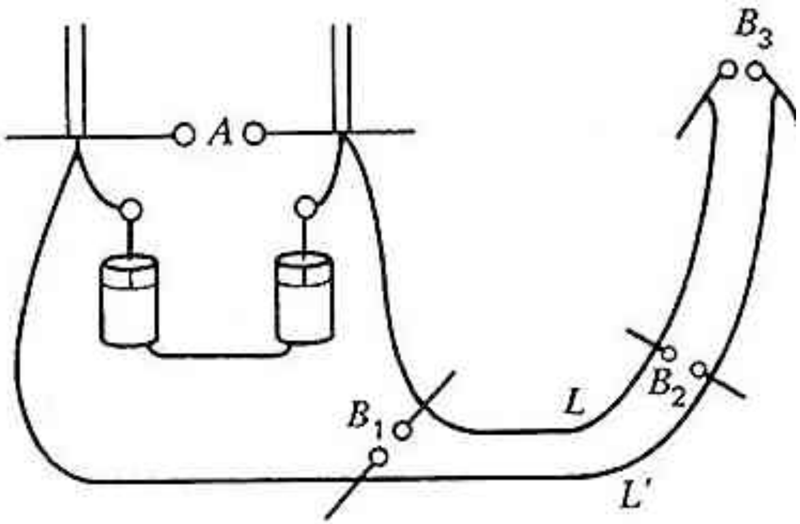
"..what a singular insight into the intricacies of the subject, and what a masterly grasp of a most difficult theory, are to be found among the writings of Mr. Oliver Heaviside. "

"self-induction ought to be a primary consideration in the design of lightning conductors..."

- *But*: Weak analogy: Discharges from Leyden jars oscillate, but lightning discharges do not.

- *However*: Leyden jar experiments lead to observation of electromagnetic waves surrounding wire conductors.

- *Feb. 1888*. Lodge notices that discharge of a Leyden jar through a short wire causes large sparks to jump a gap between free ends of wires attached to jar and extending some distance away from it.



- *Discharging Leyden jars produce oscillating current at A.*
- *Oscillations at A produce EM waves that travel along wires L, L′ and reflect at far end.*
- *Incident and reflected waves at B_3 produce voltage twice as large as at A.*
- *Discharge at B_3 is most intense when path lengths L, L′ are half-integer multiples of frequency of oscillations.*

- *July 1888*. Lodge submits paper "On the Theory of Lightning Conductors" to *Philosophical Magazine*.

"[Condenser discharges]... disturb the surrounding medium and send out radia-tions, of the precise nature of light..."

- Prepares to present results at September meeting of BAAS in Bath...