07. Fresnel's Unification.

Buchwald (1989), Chaps 10-12.

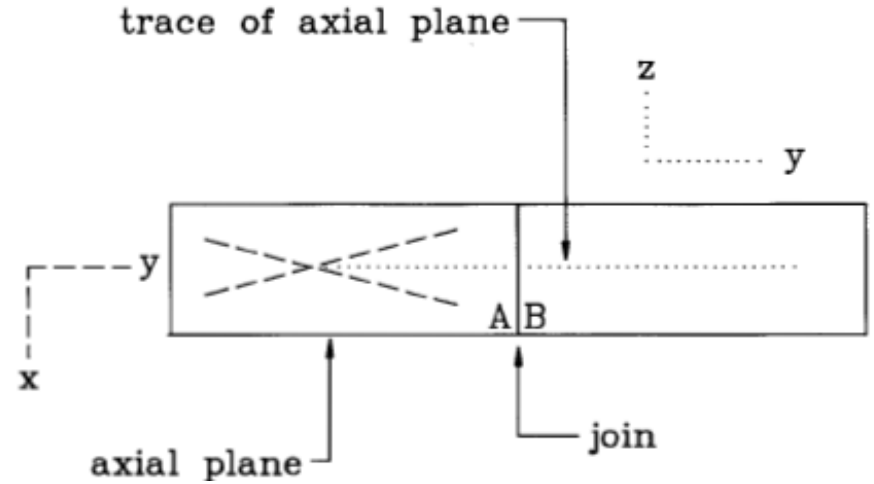
A. The Link between Polarization and Double Refraction

- *Optic axis* = direction in a crystal in which a ray does not doubly refract.
- *Uniaxial crystal* = crystal that contains one optic axis.
- *Biaxial crystal* = crystal that contains two optic axes.
- *Ordinary ray* = ray in doubly refracting crystal that has a velocity that does not vary with direction (obeys Snell's law).
- *Extraordinary ray* = ray in doubly refracting crystal that has a velocity that varies with direction (does not obey Snell's law).

- *Fresnel claims (1821)*: In *biaxial* crystals, there is no such thing as an *O* ray.
- *Theoretical explanation*: The direction of oscillation/polarization must determine the velocity of propagation (*which means there's no such thing as a ray whose speed is independent of its direction*).

  - *Why*? Light propagates because of elastic reactions to shears in the ether.
  - *And*: Unlike compressions, shears have directions.
  - *So*: If the reactions to shears vary in intensity with their directions, so must the velocity of propagation of the wave.

Experimental Confirmation:



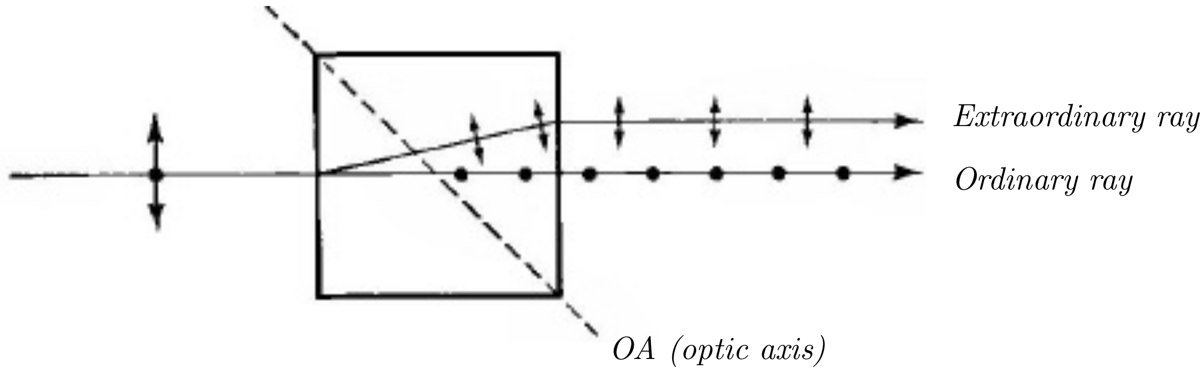
- Glue two pieces of biaxial topaz together such that piece A is cut by planes parallel to axial plane containing optic axes, and piece B is cut by planes normal to it.

- Place composite over double slits:
  (a) When either A or B covers both slits, a single interference pattern occurs.
  (b) When A covers one slit and B the other, two patterns occur: one occupies same position as in (a); the other is displaced.

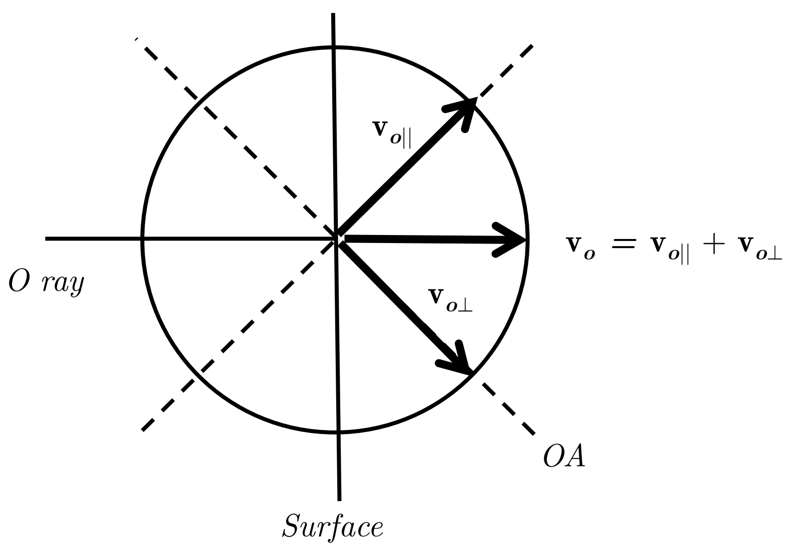
- *Now*: Determine which are the *"O"* rays.
  - *Biot's earlier result*: Plane of polarization of *"O"* rays in biaxial crystal contains the ray and the line that bisects optic axes.

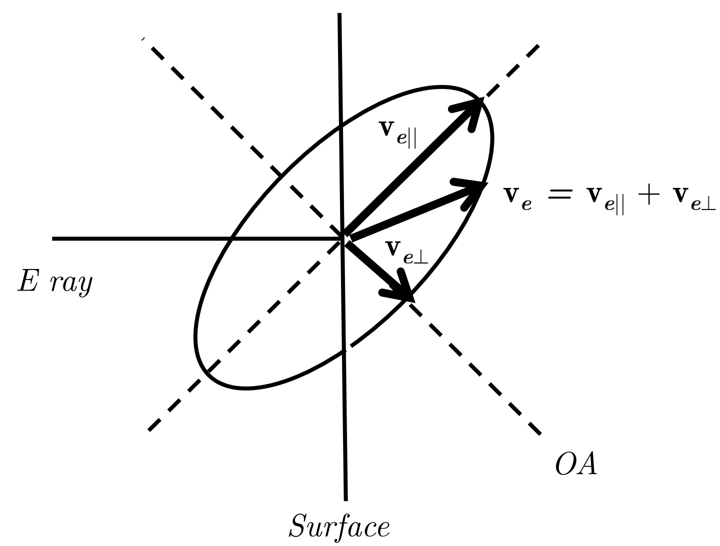
- *So*: *"O"* rays must be polarized along normal to join.
- *Fresnel observes*: Displaced pattern is polarized along normal to join.
- *So*: Displaced pattern is formed by interference of *"O"* rays.
- *Thus*: *"O"* rays in A must have different speeds than *"O"* rays in B (in order for them to interfere.)
- *Which means*: An *"O"* ray in topaz has a speed that depends on its direction.

Connection between Polarization and Velocity in Uniaxial Crystals



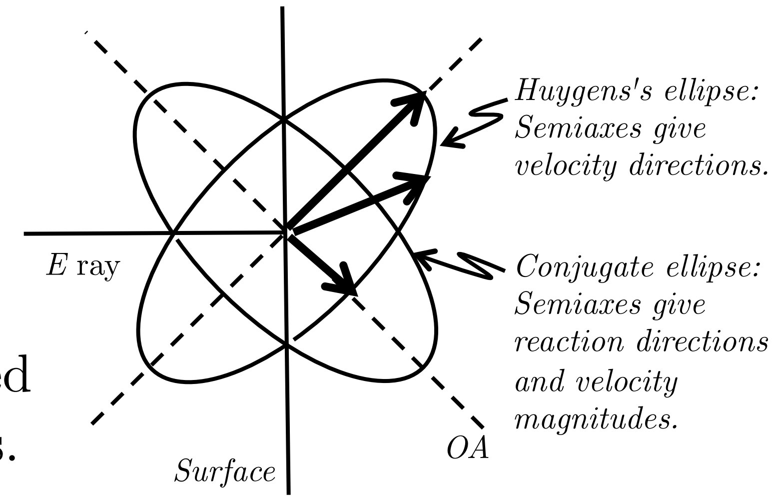
- Velocity of O ray consists of components along radii of Huygens's sphere.
- Velocity of E ray consists of components along semiaxes of Huygens's ellipse.





- *Polarizations*: O rays polarized in plane containing optic axis and ray; E rays polarized in plane normal to this.
- *Oscillations*: O oscillations are in plane normal to O polarization plane; E oscillations are in plane of O polarization.
- *Velocities*: Velocities are normal to oscillations (transversality).

- *So*: For rays in principal section, reactions due to E rays are given by semiaxes of an ellipse ("conjugate ellipse") rotated 90° from Huygens's ellipse and in same plane.



- ***Ellipsoid of elasticity*** = ellipsoid generated by rotating conjugate ellipse about optic axis.

- *Conjecture*: For rays *not* in principal section, reactions are determined by semiaxes of an ellipse obtained as a cross-section of the ellipsoid of elasticity by cutting it with a plane normal to the rays.

- *So*: *To find velocity of an arbitrary ray, cut ellipsoid of elasticity by a plane that is normal to the ray and determine the cross-section's semiaxes. Their directions give directions of reactions, and their magnitudes give velocities.*

Deriving Biot's Formula for Malus-Laplace Velocity Law for Biaxial Crystals

$v_o^2 - v_e^2 = k \sin(n) \sin(m)$, n, m = angles made by O, E rays with optic axes.

- *Frenel's Method*: Cut ellipsoid with plane $z = ax + by$ normal to rays.
  - (a) Angles between plane and circular cross-sections of ellipsoid of elasticity correspond to n, m.
  - (b) Semiaxes of cross-section of ellipsoid with plane determine speeds v_o, v_e.

> *Why (a)?* "Since the radii of the ellipsoid of elasticity determine the speeds of rays that are normal to them, it follows that the optic axes must be normal to the *circular cross-sections* of the ellipsoid -- since then the ray speeds in these two directions will not depend upon the direction of oscillation." (Buchwald, pg. 269.)

- *How Fresnel calculates (b)*: Find radius r of sphere (degenerate ellipse) cut by the plane, and then find the *minima* and *maxima* of r.

- Let equation for ellipsoid be $fx^2 + gy^2 + hz^2 = 1$.
- *Or*: For $x = \alpha z$ and $y = \beta z$, $z^2 = \dfrac{1}{f\alpha^2 + g\beta^2 + h}$.
- *So*: Equation for sphere is

$$r^2 = x^2 + y^2 + z^2 = (1 + \alpha^2 + \beta^2)z^2 = \frac{1 + \alpha^2 + \beta^2}{f\alpha^2 + g\beta^2 + h}$$

- *So*: Reciprocal of r^2, call it t, satisfies

$$f\alpha^2 + g\beta^2 + h = t(1 + \alpha^2 + \beta^2)$$

- The cutting plane satisfies $\alpha a + \beta b = 1$, so $d\beta/d\alpha = -a/b$.
- *Now*: Use results obtained from (a) to derive:

$$(1/t_{min}) - (1/t_{max}) = (f - g)\sin(n)\sin(m)$$

- *Problem*: "Recall that the semiaxes of a section cut in Fresnel's ellipsoid by a plane normal to the *ray* must be parallel to the directions of the corresponding oscillations. Yet this cannot in general be correct, because the oscillations must occur in the front proper, and this is usually oblique to the ray. Indeed, without this obliquity there could never be any divergence within a crystal between the refractions." (Buchwald, pg. 274.)

- *In other words*: Fresnel conflates *wave* speed with *ray* speed.
- *So*: Oscillations must be in plane normal to wave front, instead of ray.
- *And*: Ellipsoid must be replaced by another surface:

"...it will still be a surface of revolution that gives the laws of elasticities; but its generating curve, instead of being an ellipse, will be a curve of the fourth degree."

B. Fresnel's Surface of Elasticity

- Section Huygens's ellipsoid by xy plane with $Ox =$ optic axis.
- This yields an ellipse in principal section:

$$a^2x^2 + b^2y^2 = a^2b^2$$

- Tangent at (x', y') is given by:

$$\left.\frac{d}{dx}\right|_{(x',y')}\left(a^2x^2 + b^2y^2\right) = 2a^2x + 2b^2y\left.\frac{dy}{dx}\right|_{(x',y')} = 0$$

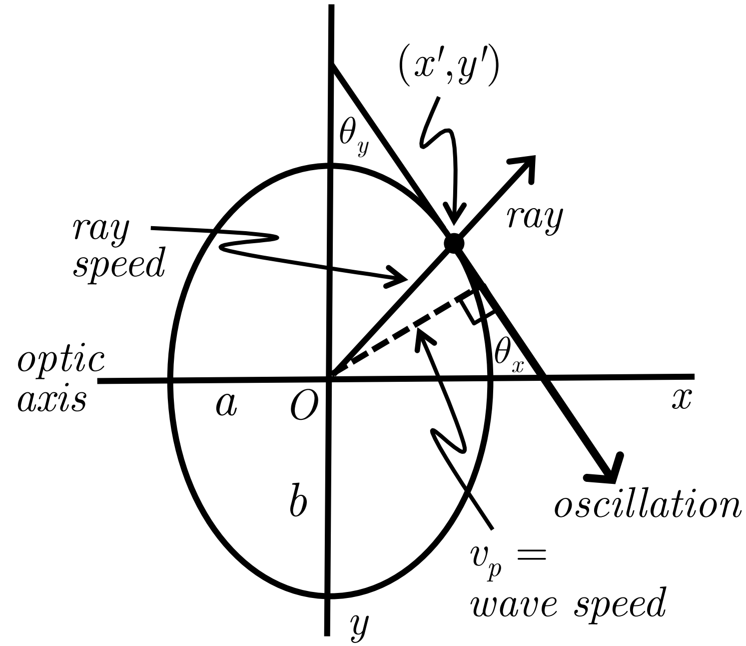
- So: $\left.\dfrac{dy}{dx}\right|_{(x',y')} = -\dfrac{a^2x'}{b^2y'}$

- Thus: Equation of tangent line is

$$y - y' = (-a^2x'/b^2y')(x - x')$$

- Or:

$$(a^2x')x + (b^2y')y - [a^2x'^2 + b^2y'^2] = 0$$



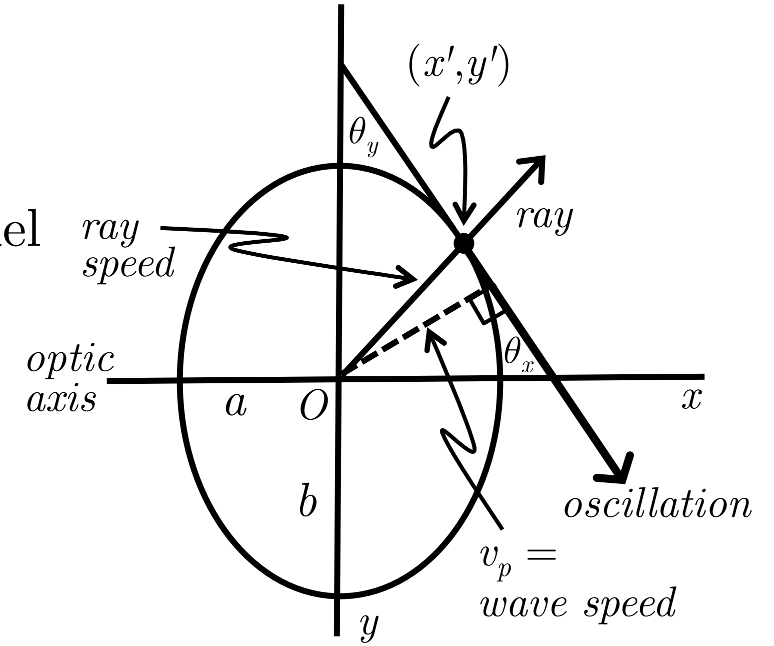
Distance d between point (x_0, y_0) and line $Ax + By + C = 0$ is

$$d = \left|Ax_0 + By_0 + C\right| / \sqrt{A^2 + B^2}$$

- So: *Distance squared from O to tangent line is:*

$$\boxed{v_p^2 = \frac{(a^2x'^2 + b^2y'^2)^2}{b^4y'^2 + a^4x'^2} = \frac{a^4b^4}{b^4y'^2 + a^4x'^2}}$$

$$\boxed{v_p^2 = \frac{(a^2x'^2 + b^2y'^2)^2}{b^4y'^2 + a^4x'^2} = \frac{a^4b^4}{b^4y'^2 + a^4x'^2}}$$



- Surface of elasticity must have semiaxis parallel to tangent line and magnitude equal to v_p.

- *Note*: $\cos^2\theta_x = \dfrac{1}{1 + \left[\dfrac{dy}{dx}\right]^2} = \dfrac{b^4y'^2}{b^4y'^2 + a^4x'^2}$

- *And*: $\sin^2\theta_x = \dfrac{a^4x'^2}{b^4y'^2 + a^4x'^2}$

- *So*: $a^2\cos^2\theta_x + b^2\cos^2\theta_y = v_p^2$

"...describes the curve that results by sectioning the true surface of elasticity by any plane that contains the optic axis."

- *Now*: Rotate about optic axis Ox to get $r^2 = a^2\cos^2\theta_x + b^2(\cos^2\theta_y + \cos^2\theta_z)$.

- *More generaly*:

Fresnel's Surface of Elasticity

$$r^2 = a^2x^2 + b^2y^2 + c^2z^2$$

C. Fresnel's Wave Surface

- Surface of elasticity gives wave speeds, not ray speeds.
- *Task*: Find "wave surface" that gives ray speeds.
- *Characteristics*: "Must be... the locus after a unit time of all the plane waves that are emitted initially from a given point, because the rays are drawn from the point to the surface and the fronts are the surface's corresponding plane tangents." (Buchwald, pg. 281.)
    - *For a given wave front in a given direction, can use surface of elasticity to calculate its wave speeds, and thus its position at a given time.*
    - *Do the same thing for wave fronts in all directions to get wave surface at a given time.*
    - *Reduces to the calculation of an envelope surface.*

En route to the Wave Surface: Fresnel's Normal Surface

- *Task*: Construct a surface with *radii* that determine the speeds of wave fronts that are normal to them.
- *Assume*: For given orientation, the velocities of a wave take extremal values.
- *So*: Need to calculate the *extrema* of the surface's *radii* in a given plane.

Surface of elasticity ($x = \alpha z,\ y = \beta z$):

$$r^2(1 + \alpha^2 + \beta^2) = a^2\alpha^2 + b^2\beta^2 + c^2$$

Start with surface of elasticity, with radii equal to wave speeds in their directions.

- *Now*: Differentiate with respect to α and set $dr = 0$.

$$\frac{d}{d\alpha}\left[r^2(1+\alpha^2+\beta^2)\right] = 2r\frac{dr}{d\alpha}(1+\alpha^2+\beta^2) + r^2\left(2\alpha + 2\beta\frac{d\beta}{d\alpha}\right) = 2r^2\left(\alpha + \beta\frac{d\beta}{d\alpha}\right)$$

$$\frac{d}{d\alpha}(a^2\alpha^2 + b^2\beta^2 + c^2) = 2\left(a^2\alpha + b^2\beta\frac{d\beta}{d\alpha}\right)$$

- *Thus*: $r^2\left(\alpha + \beta\dfrac{d\beta}{d\alpha}\right) = a^2\alpha + b^2\beta\dfrac{d\beta}{d\alpha}$

- *So*: $r^2\left(\alpha + \beta\dfrac{d\beta}{d\alpha}\right) = a^2\alpha + b^2\beta\dfrac{d\beta}{d\alpha}$

- *Now*: Let the plane be given by

  $z = mx + ny$, or $1 = m\alpha + n\beta$, or $\beta = (1 - m\alpha)/n$

- *Then*: $d\beta/d\alpha = -m/n$.
- *Thus*: $r^2(\alpha n - \beta m) = a^2\alpha n - \beta m$.

- *Or*: $\beta = \dfrac{(a^2 - r^2)n}{(b^2 - r^2)m}\alpha = \dfrac{(a^2 - r^2)n}{(b^2 - r^2)m^2}(1 - n\beta)$

- *So*: $$\boxed{\beta = \frac{(a^2 - r^2)n}{(b^2 - r^2)m^2 + (a^2 - r^2)n^2}, \quad \alpha = \frac{(b^2 - r^2)m}{(b^2 - r^2)m^2 + (a^2 - r^2)n^2}}$$

- *Now*: Substitute α, β back into equation for surface of elasticity and get:

Fresnel's Normal Surface

$$(a^2 - r^2)(c^2 - r^2)n^2 + (b^2 - r^2)(c^2 - r^2)m^2 + (a^2 - r^2)(b^2 - r^2) = 0$$

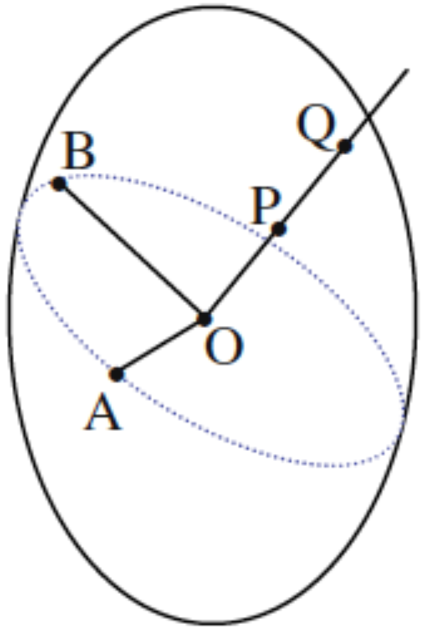
- *<u>Now</u>*: For wave surface with *radii* equal to *ray speeds* in their directions, do same thing, but start with "ellipsoid of ray speeds":

<u>Fresnel's Ellipsoid of Ray Speeds</u>

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

- ○ *Axes a, b, c are same as those of surface of elasticity.*

- *<u>Now</u>*: Cut ellipsoid by a plane through its center.
- On perpendicular to plane, mark the two points P, Q whose distances from the center, OP, OQ, are equal to the semi-axes of the elliptical cross-section, OA, OB.
- As orientation of plane changes, points P, Q trace out the wave surface:



<u>Fresnel's Wave Surface</u>

$$(a^2x^2+b^2y^2+c^2z^2)r^2-a^2(b^2+c^2)x^2-b^2(a^2+c^2)y^2-c^2(a^2+b^2)z^2$$
$$+\,a^2b^2c^2=0$$

"To sum up, Fresnel's theory involves three surfaces, the *surface of elasticity* which yields the velocity of plane waves as a function of the direction of vibration, the *index surface* [*normal surface*] which yields this velocity as a (double-valued) function of the orientation of the wave planes, and the *wave surface* ('surface de l'onde', now often called 'ray surface') which yields the expansion of a pulse from a point-like origin." (Darrigol 2010, pg. 138.*)

- *On rays versus waves*:

"The word *ray*, in the wave theory, must always be applied to the line that goes from the center of the wave to a point of its surface, whatever might be the inclination of this line to the element it ends at... So, when one wishes to translate the results of the first theory into the language of the second, one must always suppose that the line traversed by the luminous molecules, in the emission hypothesis, has the same direction as the ray taken from the center of the wave to the point of its surface that one considers."

*Darrigol, O. (2010) 'James MacCullagh's ether: An optical route to Maxwell's equations?", *the European Physical Journal H 35*, 133-172.

D. The Emerging Dominance of the Wave Theory

- 1828. Herschel's review article in *Encyclopedia Metropolitana*.
- Understands fundamental concept underlying Fresnel's account of polarization:

John Herschel
(1797-1871)

"[In Fresnel's theory] a polarized ray is one in which the vibration is constantly performed in one plane, owing either to a regular motion impressed on the luminous molecule [of the medium], or to some subsequent cause acting on the waves themselves, which disposes the planes of vibration of their molecules all one way. An unpolarized ray may be regarded as one in which the plane of vibration is perpetually varying, or in which the vibrating molecules of the luminary are perpetually shifting their planes of motion..."

- *But*: On partial polarization...

"We may conceive a partially polarized ray to consist of two unequally intense portions; one completely polarized, the other not at all... we may receive [this] as a principle, that when a surface does not completely polarize a ray, its action is such as to leave a certain portion completely unchanged, and to impress on the remaining portion the character of complete polarization."

"Thus we must conceive polarization as a property or character not susceptible of *degree*, not capable of existing sometimes in a more, sometimes in a less intense state. A single elementary ray is either wholly polarized or not at all. A beam composed of many coincident rays may be partially polarized, inasmuch as some of its component rays only may be polarized, and the rest not so."

- *Incoherent*: According to Fresnel, "No 'action' can polarize one 'portion' of light and leave the other 'portion' unaffected." (Buchwald, pg. 295.)

"Herschel's case shows, I think very clearly, that the transition from *ray as object to ray as mathematical construct* was difficult in two ways: it was difficult to make the transition in the first place, and it was just as difficult to realize that one had made it after the fact." (Buchwald, pg. 296.)

- 1837. Whewell's *History of the Inductive Sciences.*

William Whewell
(1794-1866)

"When we look at the history of the emission-theory of light, we see exactly what we may consider as the natural course of things in the career of a false theory. Such a theory may, to a certain extent, explain the phenomena which it was at first contrived to meet; but every new class of facts requires a new supposition -- an addition to the machinery; and as observation goes on, these incoherent append-ages accumulate, till they overwhelm and upset the original framework. Such was the history of the hypothesis of solid epicycles; such has been the history of the hypothesis of the material emission of light."

"In the undulatory theory, on the other hand, all tends to unity and simpli-city... It makes not a single new physical hypothesis; but out of its original stock of principles it educes the counterpart of all that observation shows."

- <u>*In other words*</u>: The emission theory isn't "wrong"; rather, it has "not the character of truth". It requires adding complicated properties to original ones to save the phenomena.

- *But*: Distinction between the emission theory and selectionism.
- *Emmision theory*: except for laws of reflection and refraction, no new formulas.
- *Selectionism*: Malus's laws for partial reflection, Biot's formula for chromatic polarization.

> "Whewell's point is well taken in this sense, that the wave theory is vastly more amenable than the alternative to quantification once it is posited that polarization controls refraction. We saw... how that assumption combined with transverality almost alone led Fresnel to his first theory for biaxial crystals. Neither the emission theory's forces nor pure selectionism's asymmetries suggested any way to quantify a connection between a ray's symmetry and its speed." (Buchwald, pg. 300.)