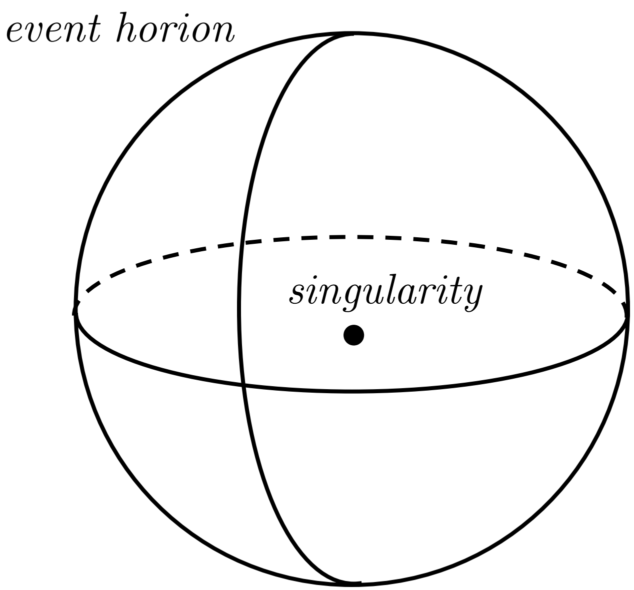
# 17. Black Hole Thermodynamics. Part 2.

## 1. Area and Entropy.

- *Question:* What happens when a physical system with a large amount of entropy is thrown into a black hole?




- *Entropy of coffee cup disappears!*
- *Violation of 2nd Law for closed system of black hole + coffee cup?*

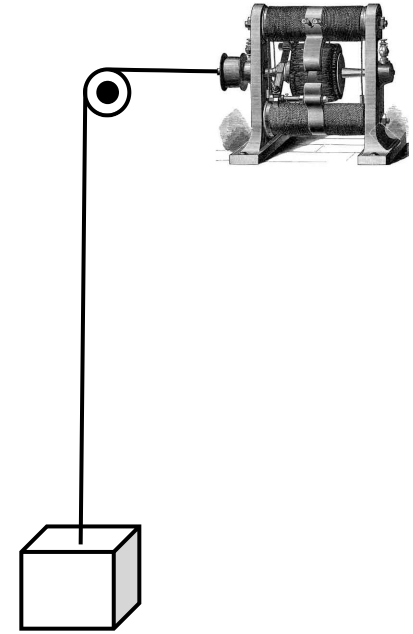
- *Bekenstein (1973):* Suppose black holes have an entropy $S_{bh}$ proportional to their area: $S_{bh} = f(A) = A/4$.

*Jacob Bekenstein*
*(Poly grad!)*

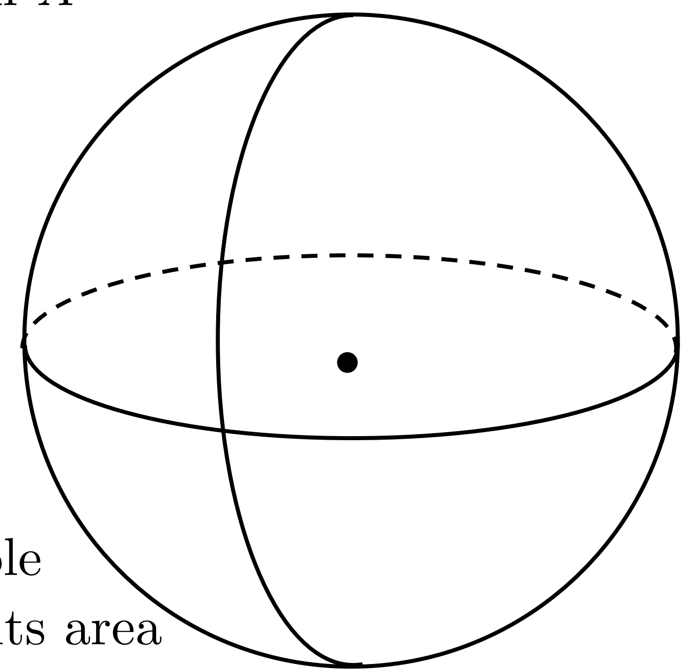
> *Generalized Second Law of Thermodynamics (GSL):*
>
> $\delta S_{bh} + \delta S \geq 0.$

Problem (Geroch 1971):

1. Lower box of radiation with high entropy toward event horizon.
2. Use weight to generate work.
3. At event horizon dump radiation in.



- At event horizon, the "Killing" vector $\xi^a$ that encodes time-translation symmetries is *null*: $|\xi^a| = 0$.
- So: At event horizon, the box has *zero energy*, $E = -\xi^a p_a$.
- So: *If box can reach horizon*, then no increase in $A$ at Step (3); thus $\delta S_{bh} = 0$.
- But: $\delta S < 0$.
- Thus: $\delta S_{bh} + \delta S < 0$. *Violation of GSL!*

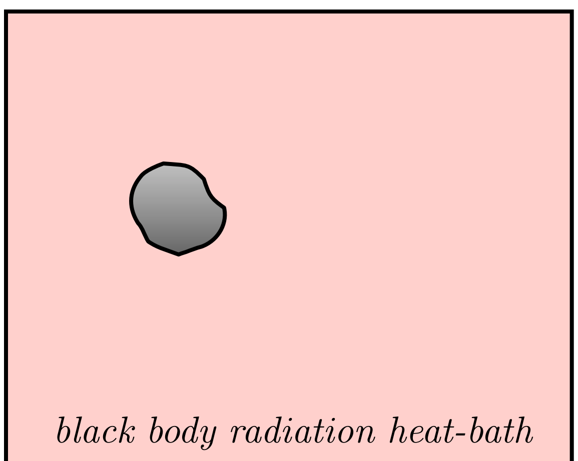


- *Bekenstein (1973)*: Box has finite size, so can't reach horizon.
- *Bekenstein (1981)*: When a stationary black hole absorbs an object with energy $E$ and radius $R$, its area increases by $\delta A = 8\pi ER$.

## 2. Surface Gravity $\kappa$ and Temperature $T$.

- *Recall*: Laws of Black Hole Mechanics look like Laws of Thermodynamics if we equate surface gravity $\kappa$ with temperature: $(1/2\pi)\kappa = T$.
- How seriously should we take this?
- *Claim A*: A black hole should be assigned *zero* absolute temperature!

> ◦ *Black body* = object that absorbs all incident radiation.
>
> ◦ *Black body radiation* = radiation emitted by a black body in thermal equilibrium.
>
> ◦ *Effective temperature* of an object = temperature of a black body that would emit the same total amount of radiation as the object.
>
> ◦ *How to measure effective temp*: Put object in thermal equilibrium with black body radiation and measure temperature of latter.



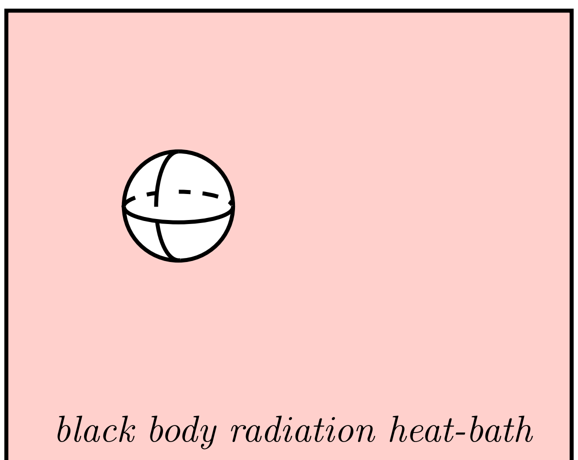

- *Object in equilibirum with heat bath.*
- $T_{object} = T_{heat\text{-}bath}$

*Refined Claim A*: The *effective temperature* of a black hole is abs zero.

*"Proof"*: "...a black hole cannot be in equilibrium with black body radiation at any non-zero temperature, because no radiation could be emitted from the hole whereas some radiation would always cross the horizon into the black hole." (Bardeen, Carter, Hawking 1973, pg. 168.)

- *Conclusion*: "In classical black hole physics, $\kappa$ has nothing to do with the physical temperature of a black hole..." (Wald 1994, pg. 149.)
- *But*: This argument depends on quantum mechanics (black body radiation can only be characterized quantum-mechanically).

Planck's (1900) quantum-mechanical formula for energy distribution of black body radiation: $E(\nu) = h\nu/(e^{h\nu/kT} - 1)$.

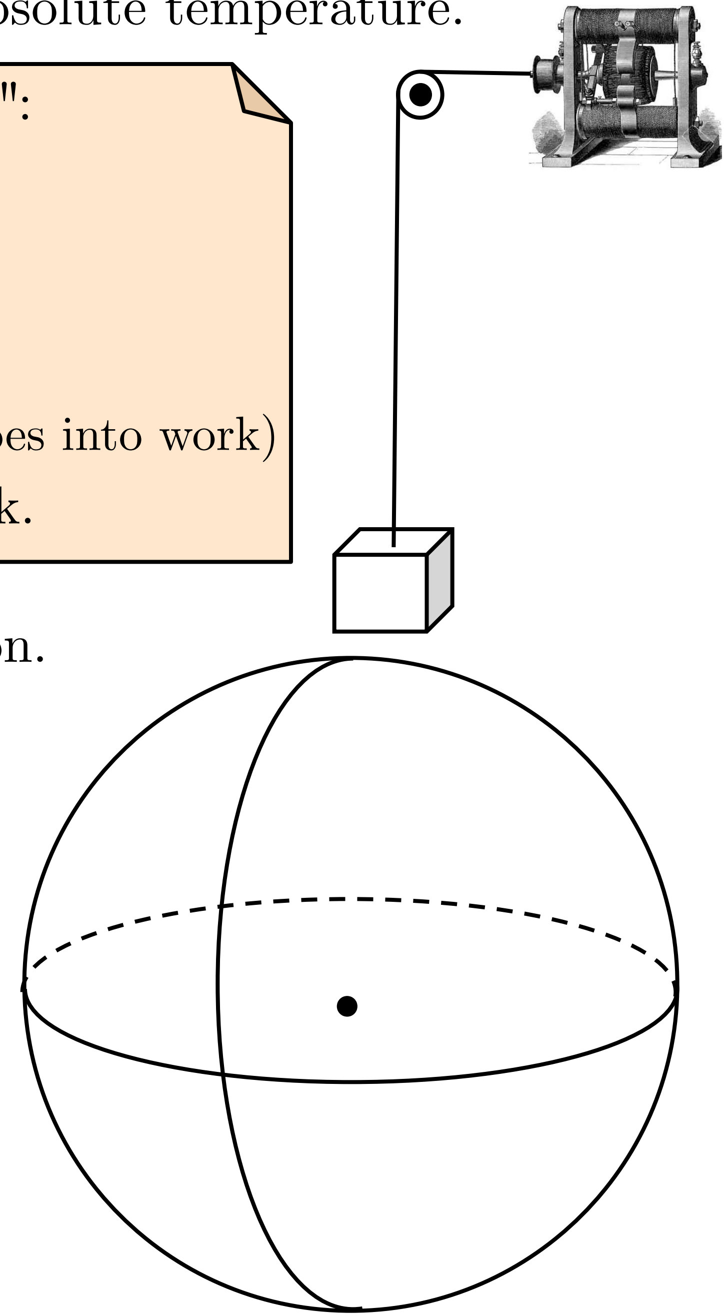



- *Black hole in heat bath.*
- *Equilibrium cannot be established.*

*Claim A*: A black hole should be assigned *zero* absolute temperature.

*Classical "Proof"*: Consider "Geroch heat engine":

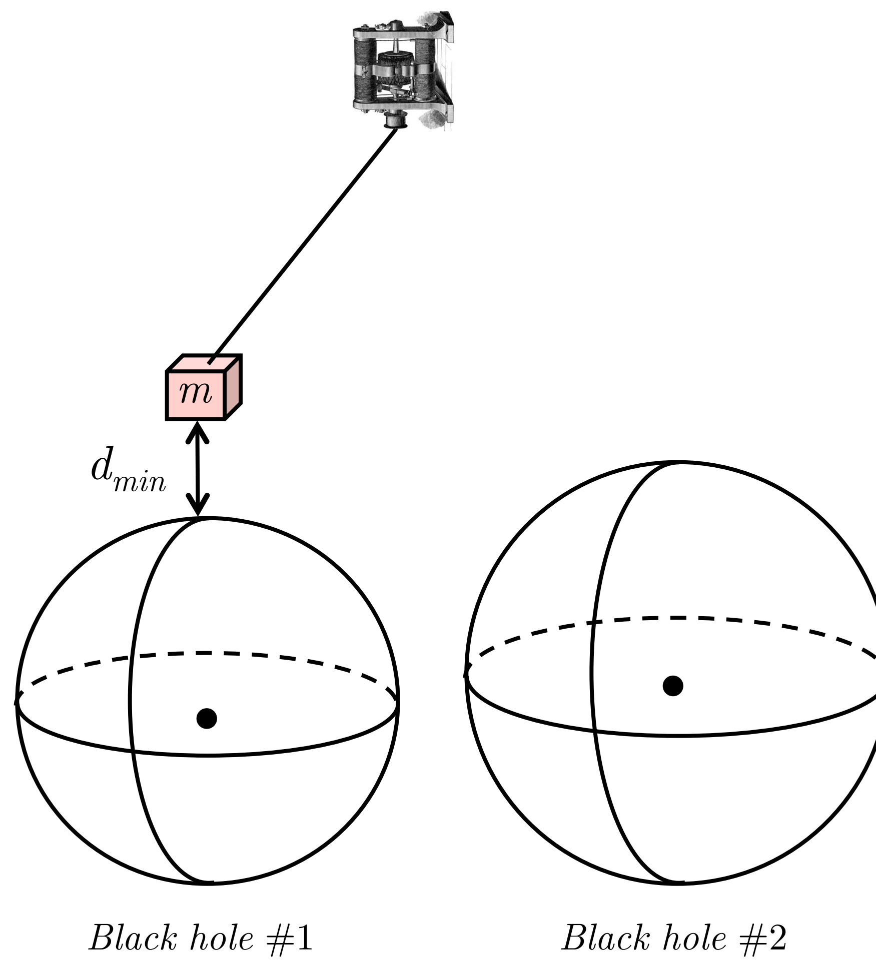
- $T_H$ = temperature of box at initial position.
- $T_C$ = temperature of black hole.
- $efficiency = W/Q_{in} = 1 - T_C/T_H$
  $= 1$ (*if* all energy of box goes into work)
- *So*: $T_C = 0$, *if* all energy of box goes into work.



- *In other words*: $T_C = 0$, *if* box can reach horizon.
- *But*: Finite box can't reach horizon.
- *Moreover*: The *ratio* $T_C/T_H$ for black holes is non-zero arbitrarily close to the horizon...

*Claim B*: $T_1/T_2 = \kappa_1/\kappa_2$ for heat engine driven by two black holes. (Jacobson 1996.)

- *Let*: $d_{min}$ = minimum distance of approach to horizon.
- *Set up black hole #1 as hot place*:
  - Lower box of radiation toward horizon of black hole #1.
  - Energy of box at $d_{min}$ is $E_1 = -\xi_1{}^a p_a = \xi_1 m$, where $\xi_1 = |\xi_1{}^a|$.
  - Raise box.



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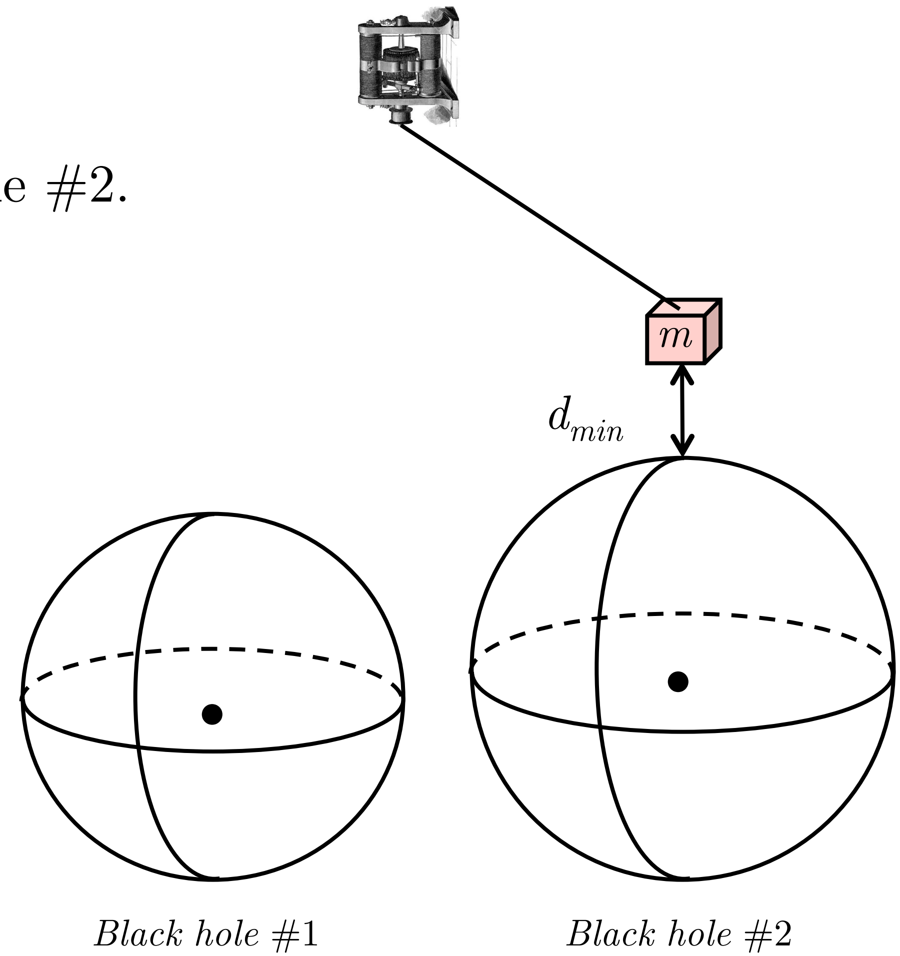
  - Raise box.
- *Use black hole #2 as cold place*:
  - Lower box toward horizon of black hole #2.
  - Energy of box at $d_{min}$ is $E_2 = \xi_2 m$.
  - Dump radiation into black hole #2.



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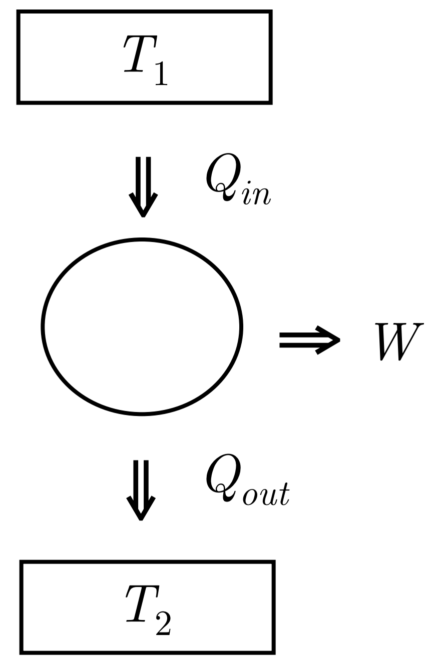
  - Dump radiation into black hole #2.

$T_1$ = temp of black hole #1.
$T_2$ = temp of black hole #2.
$Q_{in} = E_1$ = energy extracted from black hole #1.
$Q_{out} = E_2$ = energy exhausted to black hole #2.
$W = E_1 - E_2$.



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- *Use black hole #2 as cold place*:
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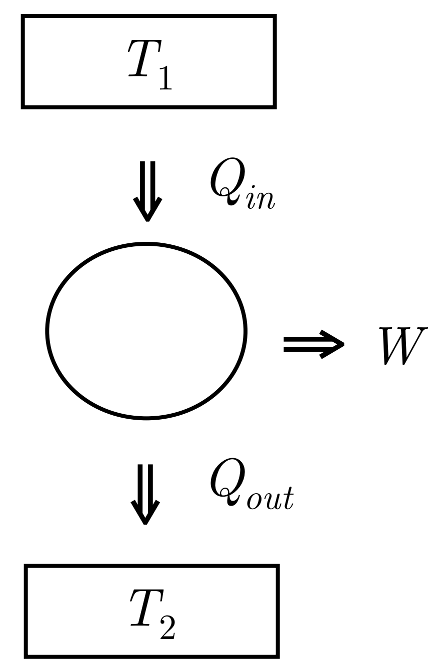



- *Now*: Define absolute temps of black holes by $T_1/T_2 := E_1/E_2 = \xi_1/\xi_2$.
- Near horizon $\xi \approx \kappa d_{min}$.

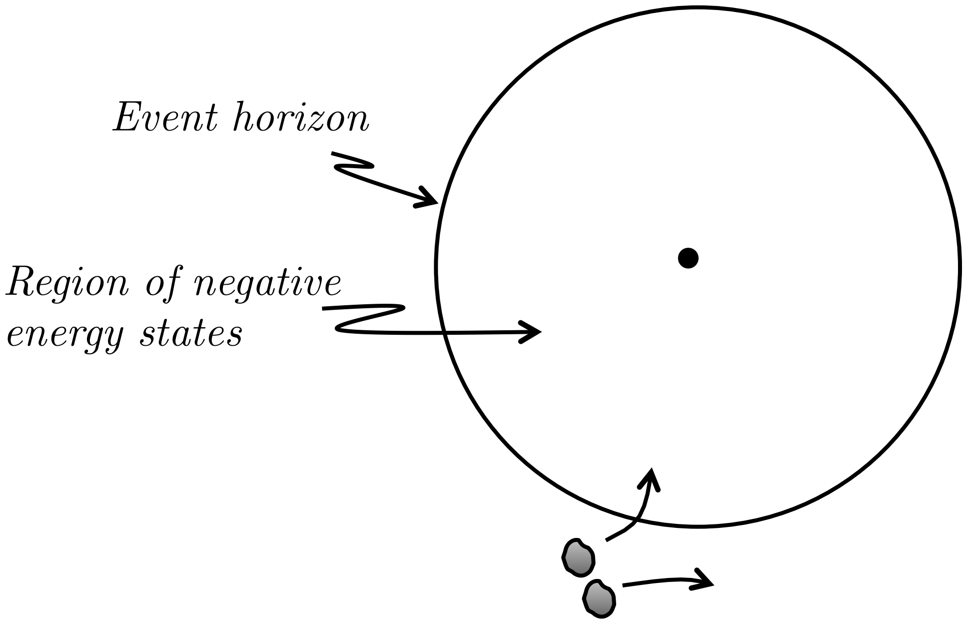
> *Why?* $\kappa = |\nabla^a \xi|$ on horizon.
> - *So*: $\xi \approx \int_0^{d_{min}} \kappa \, dx = \kappa d_{min}$

- *So*: Near horizon $T_1/T_2 = \xi_1/\xi_2 \approx \kappa_1/\kappa_2$.

### 3. Hawking Radiation.

- *Hawking (1975)*: Black holes emit radiation at the same rate that a black body would at temperature $T = (1/2\pi)\kappa$*!*

"One might picture this...in the following way. Just outside the event horizon there will be virtual pairs of particles, one with negative energy and one with positive energy. The negative particle is in a region which is classically forbidden but it can tunnel through the event horizon to the region inside the black hole where the Killing vector which represents time translations is spacelike. In this region the particle can exist as a real particle with a timelike momentum vector even though its energy relative to infinity as measured by the time translation Killing vector is negative. The other particle of the pair, having a positive energy, can escape to infinity where it constitutes a part of the thermal emission described above. The probability of the negative energy particle tunnelling through the horizon is governed by the surface gravity $\kappa$ since this quantity measures the gradient of the magnitude of the Killing vector or, in other words, how fast the Killing vector is becoming spacelike."




- *Particle/antiparticle pair production in quantum vacuum near event horizon.*
- *Negative energy antiparticle tunnels through event horizon and falls into singularity, decreasing black hole's area.*
- *Positive energy particle escapes in form of thermal radiation.*

"It should be emphasized that these pictures of the mechanism responsible for the thermal emission and area decrease are heuristic only and should not be taken too literally... The real justification of the thermal emission is the mathematical derivation..."

***Quantum field-theoretic explanation:***

*Black hole acts as scattering potential for particle states of a quantum field $\phi$.*

*Particle states in distant past*:

- Expand $\phi$ in basis $\{f_w\}$ of positive frequency solutions with respect to past: $\phi = \int d\omega \, (a_\omega \, f_\omega + \, a_\omega{}^\dagger \, f_\omega{}^*)$.
- $a_\omega{}^\dagger$, $a_\omega$ are raising/lowering operators for "in" particle states.
- "In" vacuum $|0\rangle_{in}$ = state with no "in" particles: ${}_{in}\langle 0|a_\omega{}^\dagger a_\omega|0\rangle_{in} = 0$.

*Particle states in distant future*:

- Expand $\phi$ in basis $\{p_w, \, q_w\}$, where $p_w$ are positive frequency solutions with respect to future, and $q_w$ are solutions with respect to event horizon: $\phi = \int d\omega \, (b_\omega \, p_\omega + \, b_\omega{}^\dagger \, p_\omega{}^* + \, c_\omega \, q_\omega + \, c_\omega{}^\dagger \, q_\omega{}^*)$.
- $b_\omega{}^\dagger$, $b_\omega$ are raising/lowering operators for "out" particle states.
- "Out" vacuum $|0\rangle_{out}$ = state with no "out" particles: ${}_{out}\langle 0|b_w{}^\dagger b_w|0\rangle_{out} = 0$.

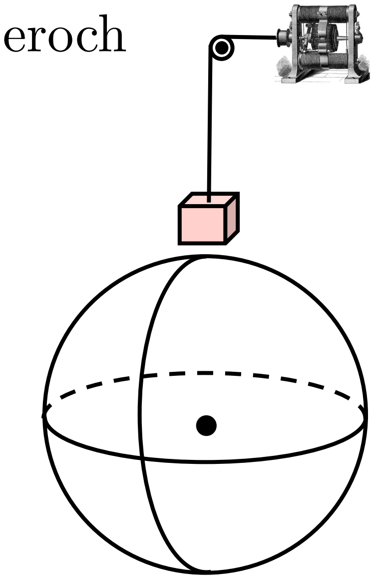
*Number of "in" particles in "out" vacuum*:

- ○ ${}_{in}\langle 0|b_w{}^{\dagger} b_w|0\rangle_{in} = \dfrac{\boldsymbol{\Gamma}_\omega}{e^{2\pi\omega/\kappa} - 1}$.
- ○ $\boldsymbol{\Gamma}_\omega$ = fraction of in-coming modes that are absorbed by black hole.
- ○ $\dfrac{1}{e^{2\pi\omega/\kappa} - 1}$ = energy distribution of black body radiation with temp $\boldsymbol{\kappa/2\pi}$.

*But*: $|0\rangle_{in}$ and $|0\rangle_{out}$ belong to *disjoint representations* of the quantum field.

- *Which means*: It's mathematically incoherent to write ${}_{in}\langle 0|b_w{}^{\dagger} b_w|0\rangle_{in}$.

*Claim (Unruh and Wald 1982)*: Hawking radiation prevents Geroch heat engine from violating Generalized Second Law.



- *Recall*: *If* box can reach horizon, then $\delta S_{bh} = 0$, $\delta S < 0$, and thus $\delta S_{bh} + \delta S < 0$. *Violation of GSL!*
- *But*: Hawking radiation generates *buoyancy* that prevents box from reaching horizon!

## 4. Entropy Bounds and The Holographic Principle.

### *A. Bekenstein Bound*

- *Idea*: Use Geroch process to derive bound on entropy of matter.
- *Claim*: When a stationary black hole absorbs an object with energy $E$ and radius $R$, its area increases by $\delta A = 8\pi ER$.

> - Increase in mass $\delta M = (\textit{energy}) \times (\textit{red-shift factor}) = E(R/4M)$.
> - $\delta A = (dA/dM)\delta M = [d(16\pi M^2)/dM]\delta M = (32\pi M)(ER/4M) = 8\pi ER$.

- *Thus*: $\delta S_{bh} = \delta A/4 = 2\pi ER$.
- *And*: Generalized Second Law now requires: $2\pi ER + \delta S \geq 0$.
- *Note*: $\delta S = (\textit{final } S) - (\textit{initial } S) = -S$.
- *So*: The entropy $S$ of any object with energy $E$ and radius $R$ cannot exceed $2\pi ER$!

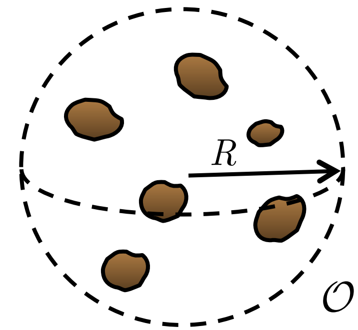
> *Bekenstein Bound (Bekenstein 1981):*
>
> $$S(X) \leq 2\pi ER$$
>
> where $E$ is the energy of an object $X$, and $R$ is the radius of the smallest sphere enclosing it.

## *B. Spherical Entropy Bound*

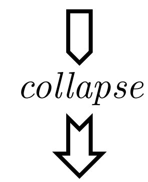
- ***Idea***: Use "Susskind" process to derive bound on entropy of matter.
- ***Suppose***: A spacetime region $\mathcal{O}$ with radius $R$ can have *more* entropy than a black hole with same radius $R$.
- ***Claim***: This would violate Generalized 2nd Law.

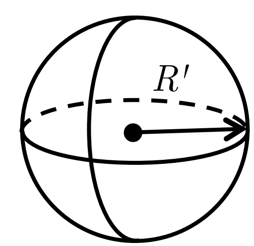
***Proof***:

- ***Let***: $S_{bh}(r)$ = entropy of black hole with radius $r$.
- ***Consider***: Process in which region $\mathcal{O}$ of radius $R$ and entropy $S > S_{bh}(R)$ collapses to form black hole with radius $R' < R$.
- ***Note***: $S_{bh}(R) > S_{bh}(R')$.
- ***So***: $S > S_{bh}(R')$.
- ***So***: $S_{bh}(R') - S < 0$.
- ***But***: $\delta S_{bh} = (\mathit{final}\ S_{bh}) - (\mathit{initial}\ S_{bh}) = S_{bh}(R')$.
- ***And***: $\delta S = (\mathit{final}\ S) - (\mathit{initial}\ S) = -S$.
- ***So***: $\delta S_{bh}(R') + \delta S < 0$. *Violation of GSL!*




*Initial state*:
$S_{total} = S > S_{bh}(R)$.



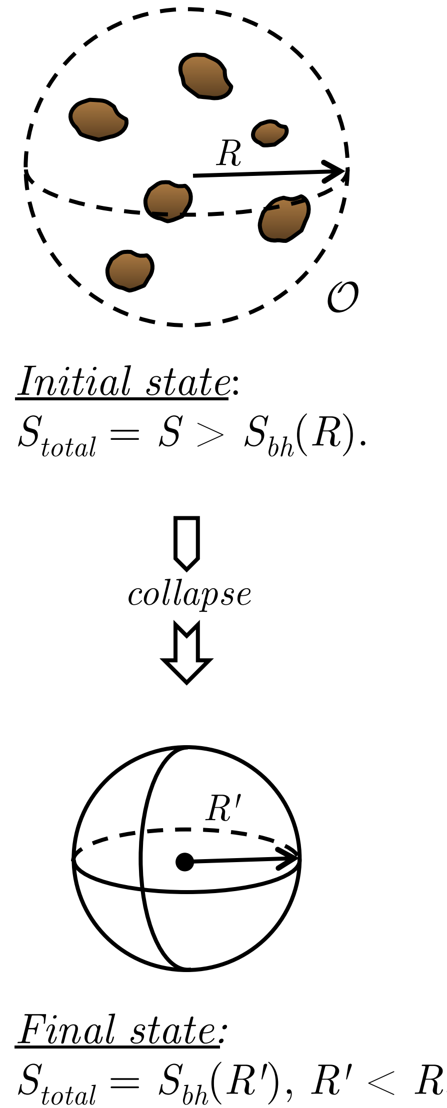




*Final state:*
$S_{total} = S_{bh}(R')$, $R' < R$.

## *B. Spherical Entropy Bound*

- ***Idea***: Use "Susskind" process to derive bound on entropy of matter.
- ***Suppose***: A spacetime region $\mathcal{O}$ with radius $R$ can have *more* entropy than a black hole with same radius $R$.
- ***Claim***: This would violate Generalized 2nd Law.
- ***So***: *If* GSL is to hold, then a region $\mathcal{O}$ with radius $R$ cannot have more entropy than a black hole with same radius $R$.

> ***Spherical Entropy Bound (Susskind 1995):***
>
> $$S(\mathcal{O}) \leq A/4$$
>
> where $\mathcal{O}$ is a spatial region with radius $R$ and $A$ is the area of a stationary black hole with radius $R$.

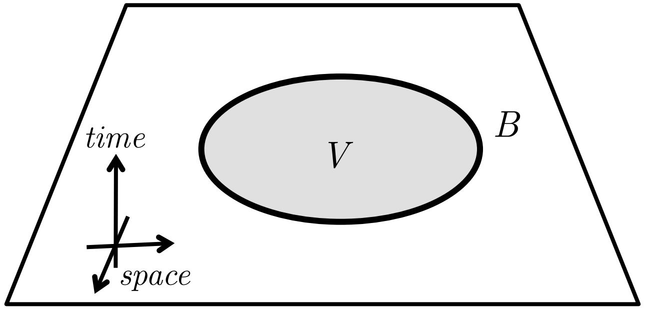


## *C. More Generalized Bounds*

*Spacelike Entropy Bound:*

$$S(V) \leq A[B(V)]/4$$

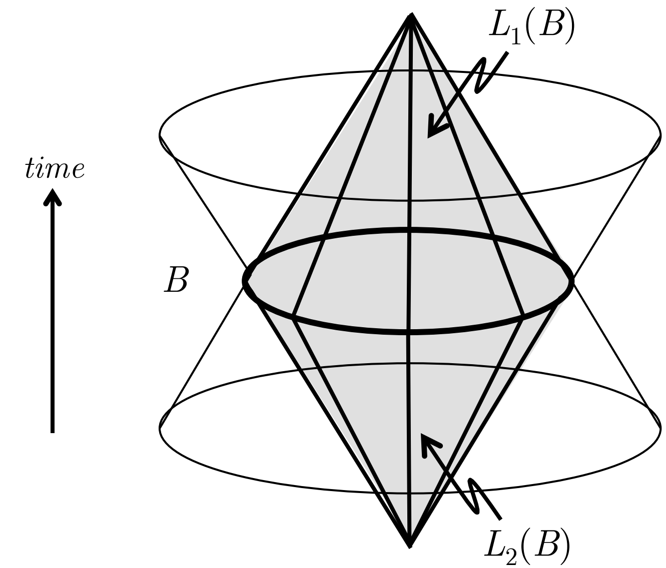
where $V$ is *any* spatial region, $B$ is its boundary, and $A$ is the area of $B$.




*Covariant Entropy Bound (Boussa 1999):*

$$S[L(B)] \leq A(B)/4$$

where $B$ is any hypersurface, $L(B)$ is any *light sheet* of $B$, and $A$ is the area of $B$.




- A *light sheet* of a surface $B$ is a null surface generated by light rays emanating from $B$ that do not expand with respect to $B$ (cross-section decreases moving outward from $B$).

- *Recall*: The Boltzmann entropy $S_B(\mathbf{\Gamma}_Z) = k \ln |\mathbf{\Gamma}_Z|$ measures the volume of the region $\mathbf{\Gamma}_Z$ in phase space associated with the macrostate $Z$.
- *Which means*: $S_B(\mathbf{\Gamma}_Z)$ is a measure of the number of microstates that have the macroproperties given by $Z$.
- *Define*: The number $N$ of *fundamental degrees of freedom* of a physical system is equal to ln (# states), which is just the system's Boltzmann entropy $S_B$.
- *Then*: The various entropy bounds suggest:

> *Holographic Principle:*
>
> The number of fundamental degrees of freedom in any region of spacetime cannot exceed $A/4$.

- *Why "holographic"?* The "information" (degrees of freedom) encoded in a physical system is contained, not in the system's volume, but in its boundary (area).

*Information-Theoretic Interpretation*

- A *degree of freedom (DOF)* = an essential property that must be assigned a value in order to characterize a state of a physical system.
- A *Boolean DOF* = an essential property that must be assigned one of two values in order to characterize a state of a physical system.
- *Let*: $\mathcal{N}$ = # of states, $N$ = # DOF, $n$ = # Boolean DOF.

> *Ex*: A spin-1/2 quantum system.
> - $N = n = 1$ (one essential property that only has 2 values)
> - $\mathcal{N} = 2$ (two possible states)

- For theories that *only* have Boolean DOF, $\mathcal{N} = 2^n$.
- *So*: For such theories, $S_B = \ln \mathcal{N} = \ln 2^n = n\ln 2$.
- *And*: $n = S_B/\ln 2 = N/\ln 2$.

> *"Information-Theoretic" Holographic Principle ('t Hooft 1995):*
>
> The number of Boolean degrees of freedom in any region of spacetime cannot exceed $A/(4\ln 2)$.

*Concerns with Holographic Principle*:

- Requires three steps:

(1) Positing a relation between (#DOF) and $\mathcal{N}$; namely, (#DOF) = $\ln \mathcal{N}$.
(2) Using this relation to identify (#DOF) with Boltzmann entropy $S_B$.
(3) Assuming the entropy $S$ of matter in the GSL is Boltzmann entropy $S_B$.

- Concern with Step (1).
  - It's motivated by Boolean theories, for which (# Boolean DOF) = $\log_2 \mathcal{N}$.
  - This suggests the generalization (#DOF) = $\log_{(\#values)} \mathcal{N}$.

*Concerns with Holographic Principle*:

- Requires three steps:

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>
> (2) Using this relation to identify (#DOF) with Boltzmann entropy $S_B$.
>
> (3) Assuming the entropy $S$ of matter in the GSL is Boltzmann entropy $S_B$.

- Concern with Step (2).
  - The (appropriate) generalization (#DOF) $= \log_{(\#values)} \mathcal{N}$ is now disanalogous with the definition of Boltzmann entropy $S_B = \ln \mathcal{N}$.
  - *Recall*: One motivation for the latter is that $S_B$ is supposed to be an *additive version* of $\mathcal{N}$:

    - The *thermodynamic entropy* of a composite system is the sum of the thermodynamic entropies of the parts: $S_{12} = S_1 + S_2$.
    - The total number of states of a composite system is the product of the total number of states of the parts: $\mathcal{N}_{12} = \mathcal{N}_1 \times \mathcal{N}_2$.

  - But (#DOF) is, supposedly, *conceptually distinct* from $\mathcal{N}$, and not *just* an additive version of $\mathcal{N}$: (#DOF) = # essential properties, $\mathcal{N}$ = # states.

*Concerns with Holographic Principle*:

- Requires three steps:

(1) Positing a relation between (#DOF) and $\mathcal{N}$; namely, (#DOF) = ln $\mathcal{N}$.
(2) Using this relation to identify (#DOF) with Boltzmann entropy $S_B$.
(3) Assuming the entropy $S$ of matter in the GSL is Boltzmann entropy $S_B$.

- Concern with Step (3).
  - Requires a "Boltzmann version" of black hole entropy $S_{bh}$.
  - *Which requires*: Identifying the microstates of a black hole and relating them to the area.
  - Some results in string theory (Strominger and Vafa 1996) and loop quantum gravity (Ashetekar, Baez, Corichi, and Krasnov 1998).