A Duality Between Topology and Entanglement? The RT Formula

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1. Review
   - *Quantum Entanglement*
   - *Entanglement Entropy*
2. Ancestors of the RT Formula
   - *Black Hole Entropy*
   - *QFT Entanglement Entropy*
   - *Topological Entanglement Entropy*
3. The RT Formula
   - *The Connectedness-Entanglement Hypothesis*
4. A Duality Between Topology and Entanglement?
   - *State-Based Approaches*
   - *Observable-Based Approaches*

1. Review

Def. 1 (*Density operator*). The **density operator** ρ for a system in one of a number of vector states $|\psi_i\rangle$, each with probablity p_i, is defined by $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\sum_i p_i = 1$.

Ex. 1. A density operator ρ can describe more than one ensemble of vector states.

$\{|a\rangle, |b\rangle; ½, ½\}$

distinct ensembles, same density operator

$\{|0\rangle, |1\rangle; ¾, ¼\}$

$$\rho = ½|a\rangle\langle a| + ½|b\rangle\langle b|$$

$$= ¾|0\rangle\langle 0| + ¼|1\rangle\langle 1|$$

where $|a\rangle = \sqrt{¾}|0\rangle + \sqrt{¼}|1\rangle$

$|b\rangle = \sqrt{¾}|0\rangle - \sqrt{¼}|1\rangle$

Why use density operators to represent states?

Def. 2 (*Pure/mixed state*). If a density operator can be expressed as $\rho = |\psi\rangle\langle\psi|$, then it is called a **pure state**. Otherwise, it is called a **mixed state**.

Standard Warning against "ignorance" interpretation of mixed states

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| + \cdots$$

Ignorance as to what vector state the system is in?

- *Concern 1*: ρ can describe more than one ensemble of vector states.
- *Concern 2*: An ignorance interpretation cannot be applied to a mixed state of a subsystem of a composite system in a pure entangled state.

Claim. ρ is a pure state *iff* $\mathrm{Tr}\,\rho^2 = 1$.
ρ is a mixed state *iff* $\mathrm{Tr}\,\rho^2 < 1$.

$\mathrm{Tr}\,(O) = \sum_i \langle w_i|O|w_i\rangle$,
where $\{|w_i\rangle\}$ is a basis for $\mathcal{H}$

Entanglement involves multipartite states!

Def. 3 (*Product/separable/non-separable density operator*).

Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ be an n-partite product space.

A **product density operator** on $\mathcal{H}$ is a density operator ρ_{prod} that can be written as the product of n terms $\rho_{\text{prod}} = \rho_1 \otimes \cdots \otimes \rho_n$, where ρ_k is a density operator on $\mathcal{H}_k$.

A **separable density operator** on $\mathcal{H}$ is a density operator ρ_{sep} that can be written as a sum of product density operators $\rho_{\text{sep}} = \sum_i p_i (\rho_i^1 \otimes \cdots \otimes \rho_i^n)$, where $\sum_i p_i = 1$, and ρ_i^k is a density operator on $\mathcal{H}_k$.

A **non-separable density operator** on $\mathcal{H}$ is a density operator that is not separable.

Entanglement involves correlations...

Def. 4 (*Expectation value*). The expectation value $\langle O \rangle_\rho$ of an observable O with respect to a state ρ is given by $\langle O \rangle_\rho \equiv \mathrm{Tr}\,(\rho O)$.

Intuition:

- $\langle O \rangle_\rho$ is the average value of O in state ρ.
- Since ρ represents an ensemble $\{|\psi_j\rangle, p_j\}$ of vector states, the average value of O in state ρ is a weighted sum of the value of O in each vector state $|\psi_j\rangle$, weighted by the probability distribution p_j:

$$\langle O \rangle_\rho = \sum_j p_j \langle \psi_j | O | \psi_j \rangle$$

- *Now check*: $\mathrm{Tr}\,(\rho O) = \sum_j p_j \langle \psi_j | O | \psi_j \rangle$

Entanglement involves correlations...

Def. 5 (*Correlated observables*). Let O_A and O_B be operators on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ with identity operators I_A and I_B. Then the observables represented by O_A and O_B are **correlated** in state ρ just when

$$\langle O_A \otimes O_B \rangle_\rho \neq \langle O_A \otimes I_B \rangle_\rho \langle I_A \otimes O_B \rangle_\rho$$

<u>*Intuition*</u>: For vector states, the Born Rule entails

$$\langle \psi | O_A \otimes O_B | \psi \rangle \neq \langle \psi | O_A \otimes I_B | \psi \rangle \, \langle \psi | I_A \otimes O_B | \psi \rangle$$

if and only if

$$\mathrm{Pr}_\psi(a_i, b_j | O_A, O_B) \neq \mathrm{Pr}_\psi(a_i | O_A) \, \mathrm{Pr}_\psi(b_j | O_B), \qquad \text{for all } i, j$$

O_A and O_B are statistically dependent in state $|\psi\rangle$

Entanglement involves correlations!

Claim. Observables represented by the operators O_i appearing in a product operator $O_1 \otimes \cdots \otimes O_n$ are uncorrelated in a product density operator state, and correlated in both separable and non-separable density operator states.

Intuition (bipartite case)

$$
\begin{aligned}
\langle O_A \otimes O_B \rangle_{\rho_{\text{prod}}} &= \mathrm{Tr}\left[(\rho_A \otimes \rho_B)(O_A \otimes O_B) \right] \\
&= \langle O_A \otimes I_B \rangle_{\rho_{\text{prod}}} \langle I_A \otimes O_B \rangle_{\rho_{\text{prod}}}
\end{aligned}
$$

$$
\begin{aligned}
\langle O_A \otimes O_B \rangle_{\rho_{\text{sep}}} &= \mathrm{Tr}\left[\textstyle\sum_i p_i (\rho_i^A \otimes \rho_i^B)(O_A \otimes O_B) \right] \\
&\neq \langle O_A \otimes I_B \rangle_{\rho_{\text{sep}}} \langle I_A \otimes O_B \rangle_{\rho_{\text{sep}}}
\end{aligned}
$$

"classically" correlated (classical mixture of product states)

$$
\langle O_A \otimes O_B \rangle_{\rho_{\text{non-sep}}} \neq \langle O_A \otimes I_B \rangle_{\rho_{\text{non-sep}}} \langle I_A \otimes O_B \rangle_{\rho_{\text{non-sep}}}
$$

"non-classically" correlated

Entanglement defined!

Def. 6 (*Entangled density operator state*). A state represented by a multipartite density operator ρ is **quantum entangled** just when ρ is non-separable.

<u>*Ex. 2*</u>.

(a) $\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$

mixed, separable, exhibits classical correlations

(b) $\rho = \frac{1}{2}\{|00\rangle + |11\rangle\}\{\langle 00| + \langle 11|\}$

pure, entangled, exhibits non-classical correlations

Warning Will Robinson: Danger!

Erkenn (2015) 80:303–337
DOI 10.1007/s10670-014-9627-8

ORIGINAL ARTICLE

Some Puzzles and Unresolved Issues About Quantum Entanglement

John Earman

<u>*Four definitions of quantum entangled state*</u>

(a) Non-product state

(b) Non-separable state

(c) Hidden-variables state

(d) Bell inequality-violating state

<u>*Moreover*</u>:

"The implicit assumption in most of the philosophical literature, and a good deal of the physics literature on quantum computing and quantum information theory, that the composite system algebra has the structure of a tensor product of Type I factors is to be deplored because it neglects possibilities that need to be explored." (Earman 2015)

For our purposes:

- Stick with pure bipartite states for the most part.
- Deal with non-Type I factor algebras when the occassion arises.

Entanglement quantified...

Def. 7 (*Reduced density operator*). Let ρ_{AB} be a density operator for a composite system with subsystems A and B. The **reduced density operator** for subsystem A is defined by $\rho_A \equiv \mathrm{Tr}_B(\rho_{AB})$.

$\mathrm{Tr}_B(O_{AB}) = \sum_i \langle w_i | O_{AB} | w_i \rangle$,
where $\{|w_i\rangle\}$ is a basis for $\mathcal{H}_B$

<u>*Ex. 3*</u>. For non-product vector state $|\psi_{AB}\rangle = \sqrt{½}\{|0_A 0_B\rangle + |1_A 1_B\rangle\}$:

$$\begin{aligned}\rho_{AB} &= |\psi_{AB}\rangle\langle\psi_{AB}| \\ &= ½\{|0_A 0_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 0_A 0_B| + |0_A 0_B\rangle\langle 1_A 1_B| + |1_A 1_B\rangle\langle 1_A 1_B|\}\end{aligned}$$

pure entangled state

$$\begin{aligned}\rho_A &= \mathrm{Tr}_B(\rho_{AB}) \\ &= ½\{|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\} = ½ I_A\end{aligned}$$

mixed states

$$\begin{aligned}\rho_B &= \mathrm{Tr}_A(\rho_{AB}) \\ &= ½\{|0_B\rangle\langle 0_B| + |1_B\rangle\langle 1_B|\} = ½ I_B\end{aligned}$$

- *Ignorance interpretation of ρ_A, ρ_B suggests qubits A, B are either $|0\rangle$ or $|1\rangle$*
- *<u>But</u>: This would entail ρ_{AB} is ½$\{|0_A 0_B\rangle\langle 0_A 0_B| + |1_A 1_B\rangle\langle 1_A 1_B|\}$.*

Entanglement quantified...

Def. 8 (*Von Neumann entropy*). The **von Neumann entropy** $S_{\mathrm{vN}}(\rho)$ of a state ρ is defined by $S_{\mathrm{vN}}(\rho) \equiv -\mathrm{Tr}(\rho \log \rho)$.

Claim. $S_{\mathrm{vN}}(\rho)$ varies from zero (for a pure state) to $\log n$ for a maximally mixed state $\rho_{\max} = (1/n) I_n$, $n = \dim \mathcal{H}$.

$S_{\mathrm{vN}}(\rho)$ is a measure of the degree to which the density operator ρ is mixed.

Aside 1: $S_{\mathrm{vN}}(\rho)$ *as a measure of information compression?*

- The *Shannon Entropy*:

$$H(X) = -\sum_i p_i \log_2 p_i$$

Specifies the minimal number of bits required to encode the output of a classical information source.

  - $X = \{x_1, \ldots, x_n\}$, where x_i is a state produced by a classical information source, and p_i is a probability distribution over such states.

- The *von Neumann Entropy*:

$$S_{\mathrm{vN}}(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i p_i \log_2 p_i$$

Specifies the minimal number of qubits required to encode the output of a quantum information source.

  - $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$,where $|\psi_i\rangle$ is a vector state produced by a quantum information source, and p_i is a probability distribution over such states.

Aside 2: $S_{\text{vN}}(\rho)$ as a measure of uncertainty?

Let $-\log_2 p_i$ be "info gained" upon finding a system to be in a state drawn from a set of states with probabilites p_i.

- The *Shannon Entropy*:

$$H(X) = -\sum_i p_i \log_2 p_i$$

- $X = \{x_1, ..., x_n\}$, where x_i is a state produced by a classical information source, and p_i is a probability distribution over such states.

Expected value of information gained upon measurement of X with outcome x_i.

- The *von Neumann Entropy*:

$$S_{\text{vN}}(\rho) = -\text{Tr}(\rho \log_2 \rho) = -\sum_i p_i \log_2 p_i$$

- $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$,where $|\psi_i\rangle$ is a vector state produced by a quantum information source, and p_i is a probability distribution over such states.

Expected value of information gained upon measurement of ρ with outcome $|\psi_i\rangle$.

<u>But</u>: $S_{\text{vN}}(\rho)$ measures mixedness of ρ.

<u>And</u>: Mixedness does not necessarily entail uncertainty...

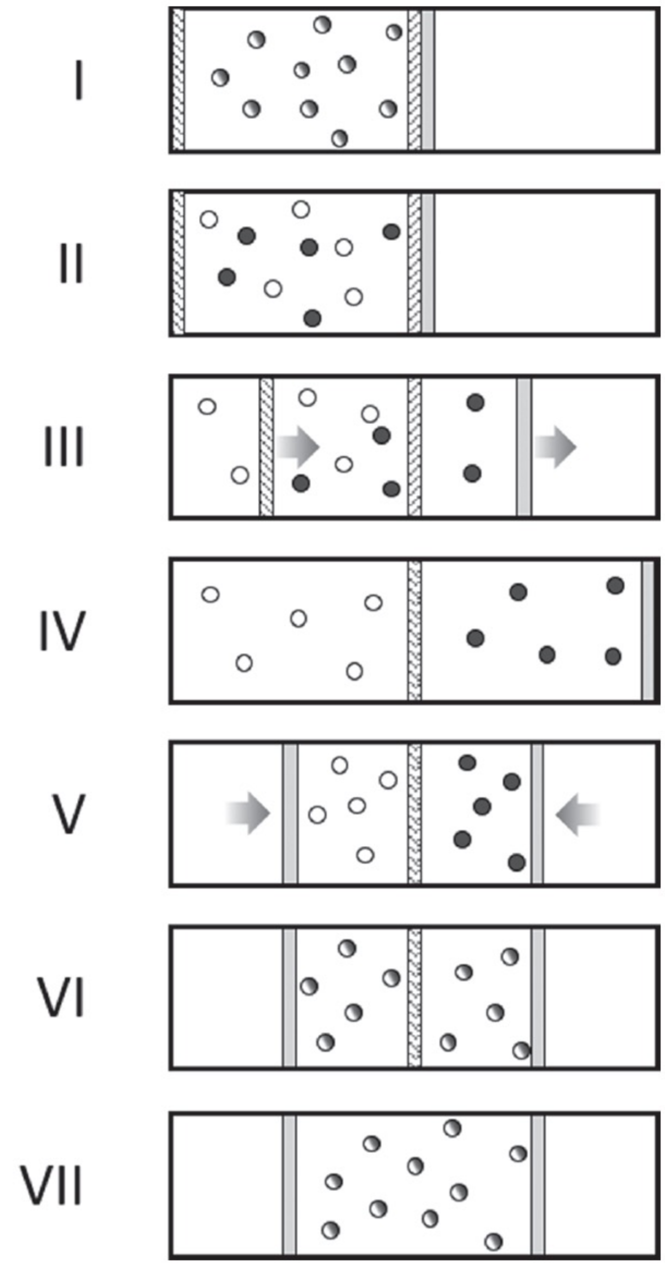
Aside 3: $S_{\mathrm{vN}}(\rho)$ as thermodynamic entropy S_{T}?

(a) Is a quantum measurement associated with an increase in S_{T}?

(b) Is a quantum measurement associated with a transition between a pure state and a mixed state?

von Neumann (1932): Yes to both!

- *Shenker (1999): Not so fast!*
- *Henderson (2003): Hold on...*
- *Hemmo & Shenker (2006): etc.*
- *Prunkl (2020)...etc.*



Prunkl (2020, pg. 265)

Entanglement quantified!

Def. 9 (*Entanglement entropy*). For a bipartite system AB with density operator ρ_{AB}, the **entanglement entropy** S_A of subsystem A is defined by:

$$S_A \equiv S_{\mathrm{vN}}(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A)$$

S_A is a measure of the degree to which the density operator ρ_A is mixed.

Claim. If ρ_{AB} is a pure state, then it is separable *if and only if*

$$S_A \leq S_{\mathrm{vN}}(\rho_{AB}).$$

- <u>*So*</u>: *If* a bipartite system is in a pure state, and *if* entanglement = non-separability, then S_A is a measure of the degree to which the density operator ρ_{AB} is entangled.

<u>More provocatively</u>: S_A is a measure of the degree to which subsystem A is entangled with subsystem B.

<u>But</u>: What about mixed states and/or n-partite entanglement, $n > 2$...

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2. Ancestors

Bekenstein-Hawking Black Hole Entropy

$$S_{\mathrm{bh}} = \mathrm{Area}(horizon)/4G$$

Bekenstein (1973)

Thermodynamical motivations

Wallace (2018)

- Small-scale interactions characterized as "reversible" and "irreversible" with horizon area A playing role of S_{T}. Christodolou & Ruffini (1971)

- Stationary black hole satisfies $dM = (1/8\pi)\kappa dA + \cdots$, where κ is surface gravity. Bardeen, Carter, Hawking (1973)

  Formally identical to $dU = TdS_{\mathrm{T}} + \cdots$, if $S_{\mathrm{T}} = A/4$ and $T = (1/2\pi)\kappa$.

- *Area theorem*
  $\Delta A \geq 0$, in finite process involving black holes. Hawking (1971)

2. Ancestors

Bekenstein-Hawking Black Hole Entropy

$$S_{\text{bh}} = \text{Area}(horizon)/4G$$

Bekenstein (1973)

Wallace (2018)

Statistical mechanical motivations

- If black holes are thermodynamical systems, they should have a statistical mechanical description, too.

$$S_{\text{Boltz}} = k \ln \Omega \qquad\qquad \Omega_{\text{bh}} = e^{A/4kG}$$

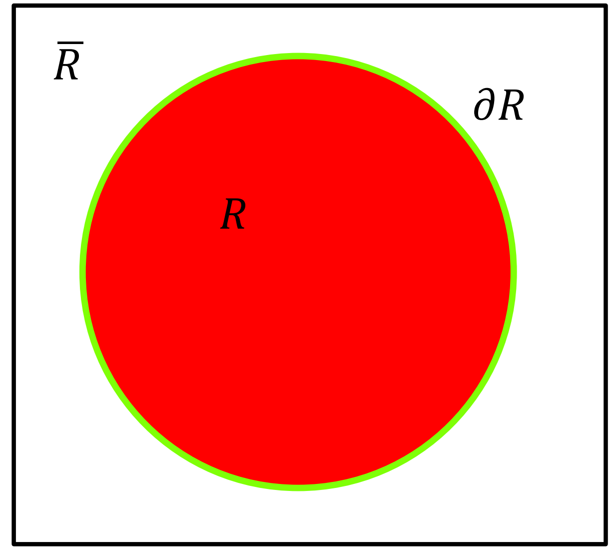
# of microstates

- *Evidence*: Calculations in EFT, string theory, AdS/CFT.

Geometric Entanglement Entropy

$$S_R = \alpha \mathrm{Area}(\partial R)$$

Degree of entanglement between DOF in R and $\overline{R}$



- S_R of ground state of free massless scalar quantum field with respect to a spherical region R of spacetime is proportional to area of boundary ∂R. Srednicki (1993)

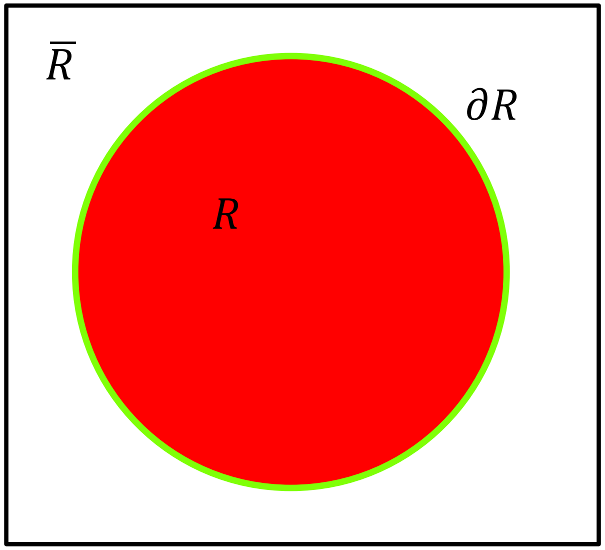
Thermodynamical motivation:

- "Entropy is usually an extensive quantity, so we might expect that $S_R \sim (\mathrm{radius})^3$.
- *But*: $S_R = S_{\overline{R}}$, which means: "...S_R should depend only on properties which are shared by the two regions...", and "The one feature they have in common is their shared boundary...".

Geometric Entanglement Entropy

$$S_R = \alpha \text{Area}(\partial R)$$

Degree of entanglement between DOF in R and $\overline{R}$



- S_R of ground state of free scalar field with respect to a conical region R of Euclidean spacetime is identical to Boltzmann entropy of Euclidean (conically singular) black hole. Callan & Wilczek (1994)

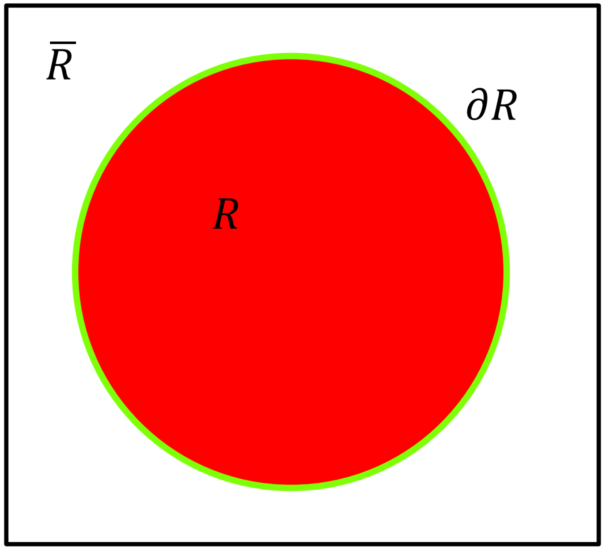
Statistical mechanical motivation:

- "...the [statistical mechanical] B-H entropy and the geometric [entanglement] entropy are the classical and first quantum contributions to a unified object measuring the response of the Euclidean path integral to the introduction of a conical singularity in the underlying geometry."

Geometric Entanglement Entropy

$$S_R = \alpha \text{Area}(\partial R)$$

Degree of entanglement between DOF in R and $\overline{R}$



S_R is *UV Divergent*

(a) Explicit calculations:

- Free energy for a scalar field diverges as black hole horizon is approached. 't Hooft (1985)
- Srednicki's result: $S_R = \alpha M \text{Area}(\partial R)$, for cut-off M.
- Callan & Wilczek's result: Divergence associated with conical singularity.

(b) Lattice-theoretic estimates: Headrick (2019, pg. 23)

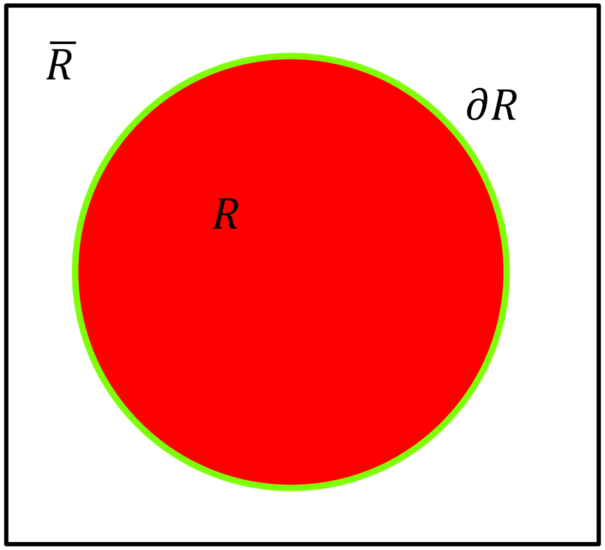
$$S_R \sim L^{D-2}/\epsilon^{D-2}$$

where L = size of region, and ϵ = lattice spacing.

__Geometric Entanglement Entropy__

$$S_R = \alpha \mathrm{Area}(\partial R)$$

Degree of entanglement between DOF in R and $\bar{R}$



__S_R is *UV Divergent*__

(c) Appeal to infinite DOF:

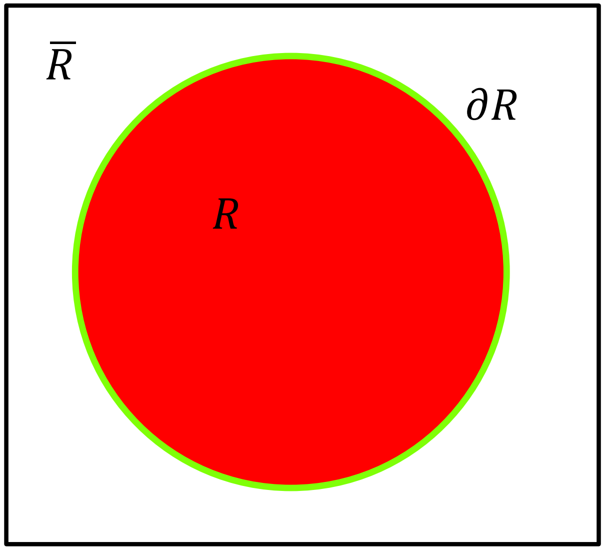
"...note that the subsystem we are talking about...actually contains an infinite number of DOF, since we have field modes with arbitrarily short wavelengths... Summing the contribution to S_R from these infinite number of modes, we obtain a divergence typically proportional to the area of the boundary..." van Raamsdonk (2017)

"In a continuum QFT there are UV modes at arbitrarily small scales across the dividing surface ∂R, and this makes it impossible to actually split the full Hilbert space." Hartman (2015)

Geometric Entanglement Entropy

$$S_R = \alpha \text{Area}(\partial R)$$

Degree of entanglement between DOF in R and $\bar{R}$



S_R is UV Divergent

(d) Appeal to non-Type I nature of local QFT algebra of observables: Swanson (2020)

No-Go Lemma
In any model of the Haag-Kastler axioms with a well-defined UV scaling limit, if the local double-cone algebras, $\mathfrak{N}(O)$, are not type I, then there is no pair of Hilbert spaces $(\mathcal{K}_1, \mathcal{K}_2)$ and an isomorphism $V : \mathcal{H} \rightarrow \mathcal{K}_1 \otimes \mathcal{K}_2$ such that $V^*\mathfrak{N}(O)V \subset B(\mathcal{K}_1) \otimes I$, with $V^*\mathfrak{N}(O')V \subset I \otimes B(\mathcal{K}_2)$.

- *Not* a consequence of the Reeh-Schlieder theorem: Swanson (2020)

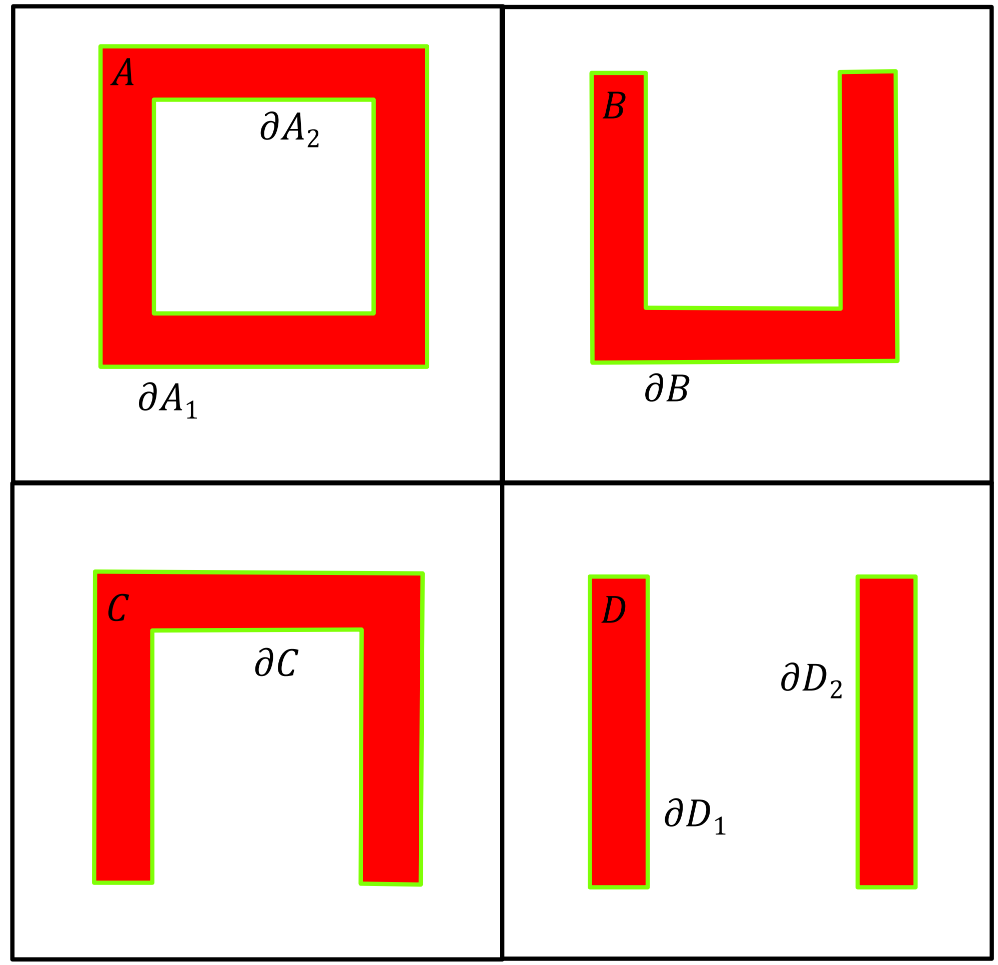
"The RS-Theorem only depends on the presence of long-range infrared correlations. In contrast, the type III property is thought to be an ultraviolet effect."

Topological Entanglement Entropy

$$S_R = \alpha \text{Area}(\partial R) - \Gamma$$

Geometric aspect
- local/short-range

Topological aspect
- global/long-range



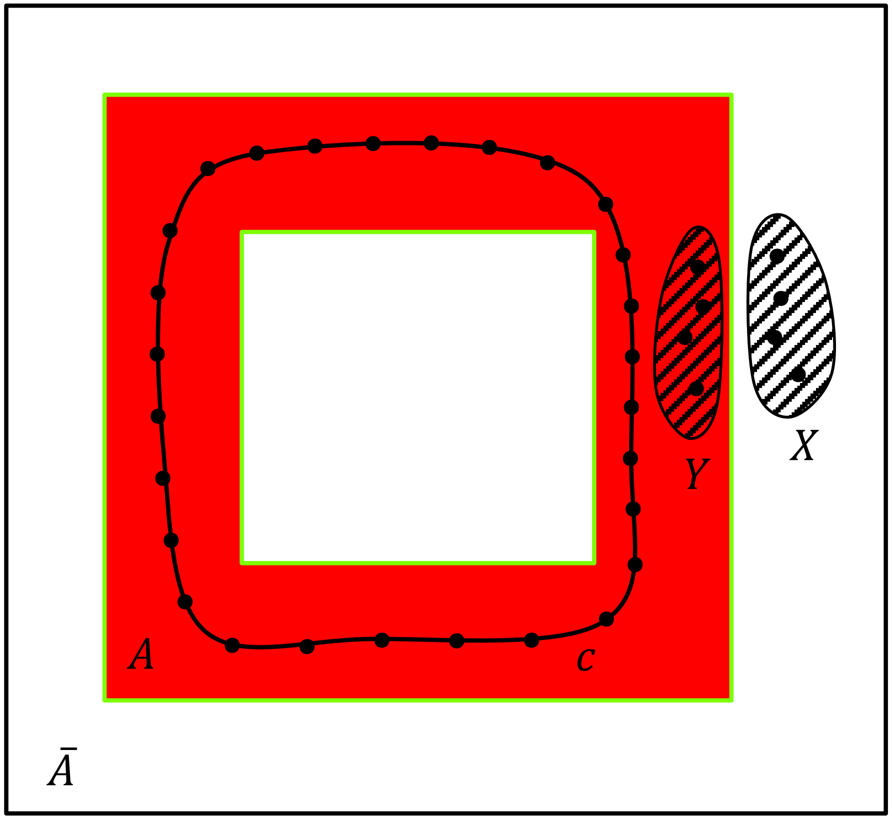
Levin & Wen (2006)

Non-zero value of Γ "signals the presence of nonlocal correlations and topological order."

Motivation

$$\begin{aligned}(S_A - S_B) - (S_C - S_D) &= \{[\alpha \text{Area}(\partial A_1) - \Gamma] + [\alpha \text{Area}(\partial A_2) - \Gamma] - [\alpha \text{Area}(\partial B) - \Gamma]\} - \\ &\quad \{[\alpha \text{Area}(\partial C) - \Gamma] - [\alpha \text{Area}(\partial D_1) - \Gamma] - [\alpha \text{Area}(\partial D_2) - \Gamma]\} \\ &= -2\Gamma\end{aligned}$$

Local vs. Non-Local Correlations?



- Regions $X \subset \bar{A}$, $Y \subset A$ are contractible sets of lattice sites.
- Loop $c \subset A$ is a non-contractible set of lattice sites.
- Let W_c be a "non-local" Wilson loop operator.
- Let O_X, O'_Y be "local" operators.

Example of non-local correlation

$\langle W_c O_X \rangle \neq \langle W_c \rangle \langle O_X \rangle$

Example of local correlation

$\langle O_X O'_Y \rangle \neq \langle O_X \rangle \langle O'_Y \rangle$

"The difference $S_A - S_B$ has contributions that come from the upper horizontal part of the region A. Similarly, non-zero contributions to $S_C - S_D$ come from the same upper part. If these entropies only had contributions from local correlations (i.e., if it was $S_X = \alpha \mathrm{Area}(\partial X)$), their difference would go to zero when the respective regions become sufficiently large. The only possible contribution could arise from a non-local operator like a Wilson loop, that wraps non-trivially around the region A."

Pachos (2012)

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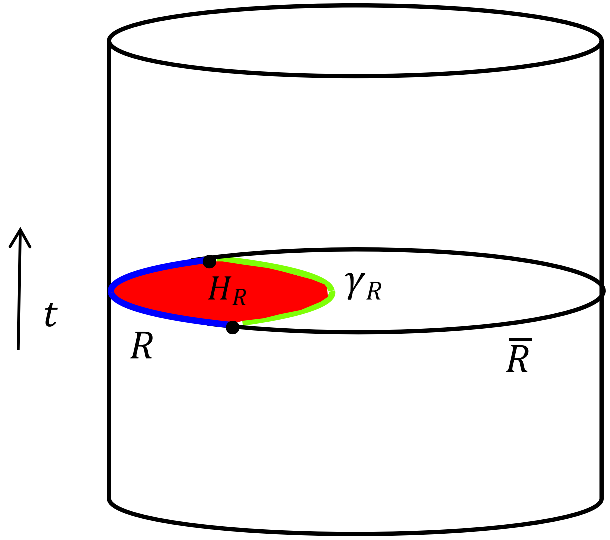
3. The RT Formula

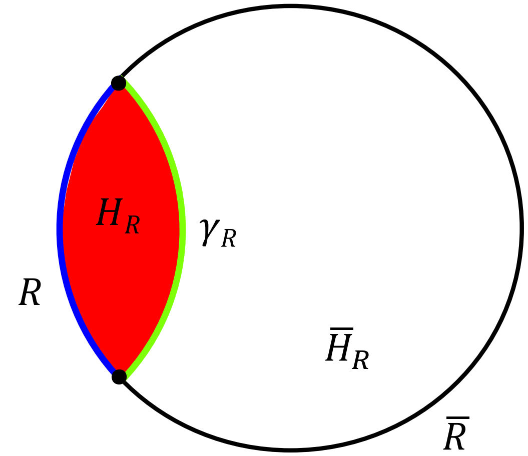
Ryu & Takayanagi (2006)

<u>*RT Formula*</u>

$$S_R = \mathrm{Area}(\gamma_R)/4G$$

Degree of entanglement between boundary systems in R and $\overline{R}$.

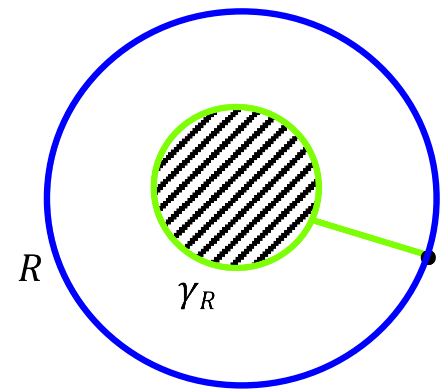




- $R \cup \overline{R}$ = boundary timeslice
- γ_R = minimal area bulk surface with same boundary as R
- H_R = homology surface bounded by $\gamma_R \cup R$

<u>*Motivations*</u>

(a) *Analogy with black hole entropy*:
For AdS-Schwarzschild spacetime, in limit in which R encompasses entire boundary timeslice, γ_R becomes event horizon.



(b) *Derivation*:
Derivable via Euclidean path integral techniques via replica trick.

Lewkowycz & Maldacena (2013)

3. The RT Formula

Ryu & Takayanagi (2006)

RT Formula

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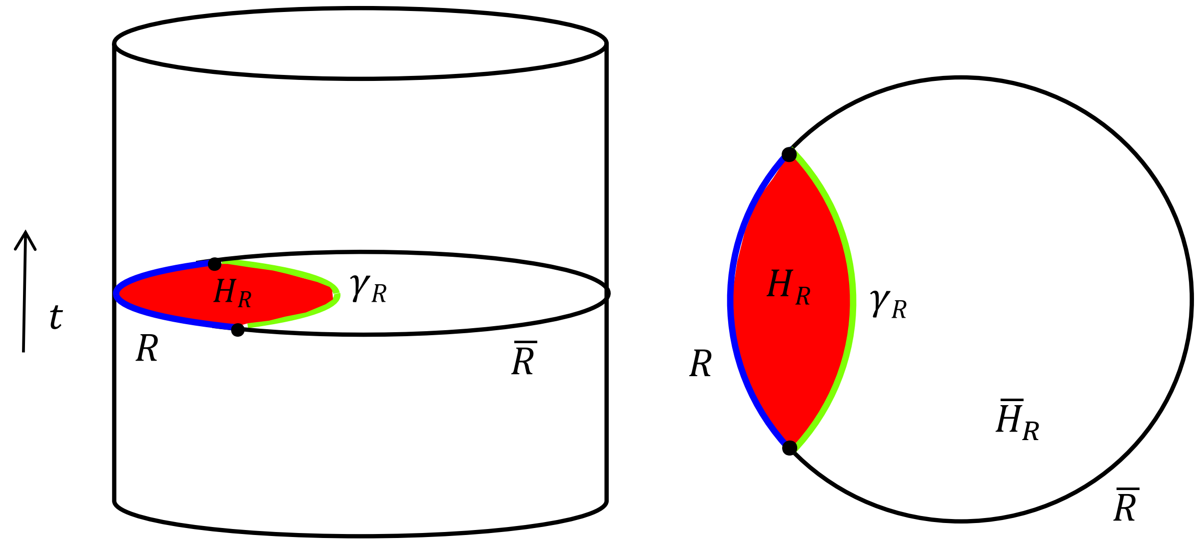
Degree of entanglement between boundary systems in R and $\overline{R}$.



- $R \cup \overline{R}$ = boundary timeslice
- γ_R = minimal area bulk surface with same boundary as R
- H_R = homology surface bounded by $\gamma_R \cup R$

Qualifications

Bulk theory must be classical, Einsteinean gravity with time-reflection symmetry underwhich R is invariant.

- Covariant version — Hubeny, Rangamani, Takayanagi (2007)
- Quantum correction — Faulkner, Lewkowycz, Maldacena (2013)

$$S_R = \mathrm{Area}(\gamma_R)/4G + S_{H_R}$$

Degree of entanglement between bulk systems in H_R and $\overline{H}_R$.

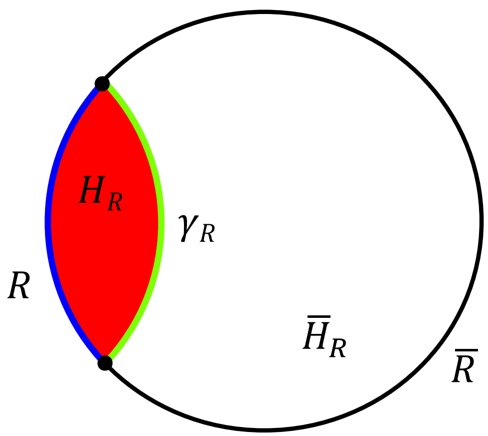
Connectedness/Entanglement Hypothesis

van Raamsdonk (2010)

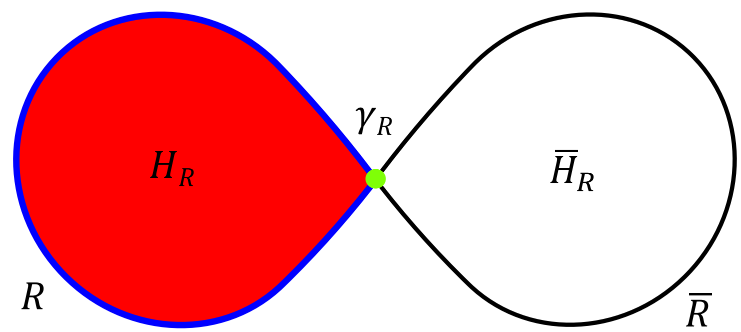
$$S_R = \text{Area}(\gamma_R)/4G$$

$$|\Psi\rangle = \sum_{i,j} p_{ij}\, |\psi_i^R\rangle \otimes |\psi_j^{\bar{R}}\rangle \xrightarrow{S_R \to 0} |\Phi\rangle = \left(\sum_i c_i |\psi_i^R\rangle\right) \otimes \left(\sum_j d_j |\psi_j^{\bar{R}}\rangle\right)$$

AdS/CFT dual $\updownarrow$ $\updownarrow$ *AdS/CFT dual*



$\xrightarrow{\text{Area}(\gamma_R) \to 0}$



> "Classical connectivity arises by entangling the degrees of freedom of the two components [of a CFT product state]".

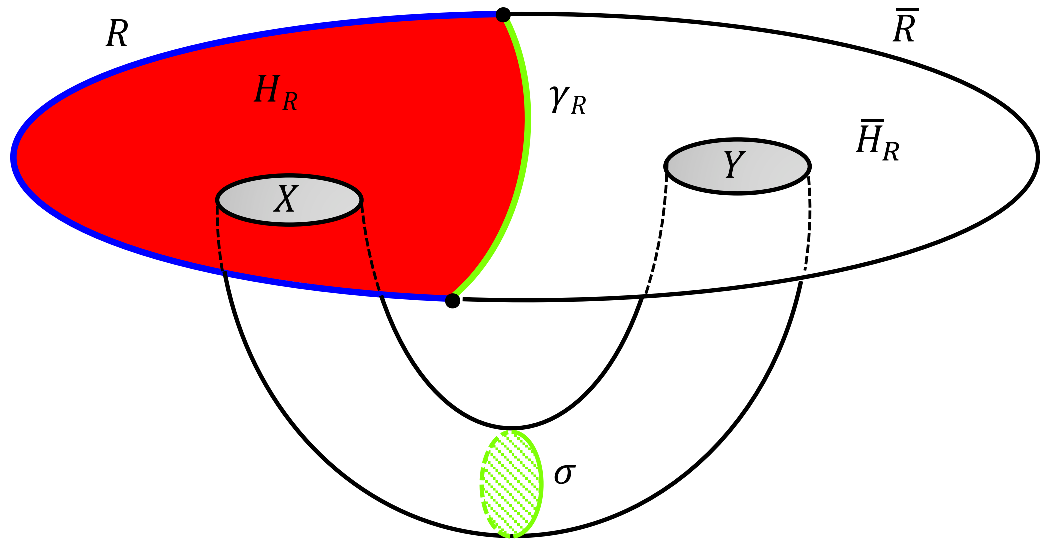
But what about the quantum correction term?

(ER=EPR)-Modified RT Formula

RT Formula with quantum correction

$$S_R = \text{Area}(\gamma_R)/4G + S_{H_R}$$

Degree of entanglement between X and Y.



Proposal: S_{H_R} is proportional to the minimal area cross-section σ of a corresponding wormhole.

$$S_R = \text{Area}(\gamma_R)/4G + \alpha\text{Area}(\sigma)$$

Connectedness/Entanglement Hypothesis:

(*bulk connectedness*) $\Rightarrow$ (*boundary entanglement*)

ER=EPR:

(*bulk connectedness*) $\Leftrightarrow$ (*boundary entanglement*)

1. Review
   - *Quantum Entanglement*
   - *Entanglement Entropy*
2. Ancestors of the RT Formula
   - *Black Hole Entropy*
   - *QFT Entanglement Entropy*
   - *Topological Entanglement Entropy*
3. The RT Formula
   - *The Connectedness-Entanglement Hypothesis*
4. A Duality Between Topology and Entanglement?
   - *State-Based Approaches*
   - *Observable-Based Approaches*

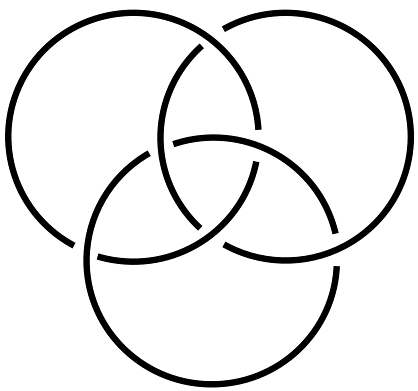
4. A Duality Between Topology and Entanglement?

State-Based Approaches

Goal: Identify topological states that are dual to quantum entangled states.

(Aravind 1997)

Example: Aravind's Analogy
Topologically entangled n-links are *analogous* to quantum entangled n-partite states.



Borromean rings 3-link

$$|\Psi\rangle_{\mathrm{GHZ}} = \sqrt{\tfrac{1}{2}}\left\{|\uparrow_{z_1}\rangle|\uparrow_{z_2}\rangle|\uparrow_{z_3}\rangle - |\downarrow_{z_1}\rangle|\downarrow_{z_2}\rangle|\downarrow_{z_3}\rangle\right\}$$

GHZ tri-partite vector state

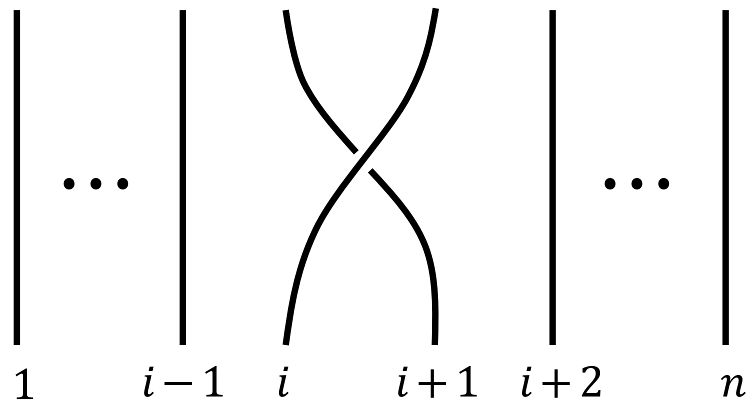
An ***n*-link** is an embedding of a disjoint collection of n circles, taken up to ambient isotopy.

Towards strengthening the analogy...

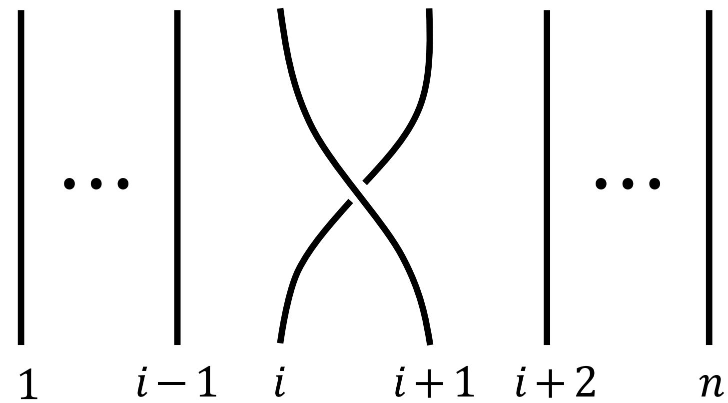
The **n-dim braid group** B_n is the group generated by $n-1$ generators $\sigma_1, ..., \sigma_{n-1}$ that satisfy

(a) $\sigma_i\sigma_j = \sigma_j\sigma_i$ $\quad\forall i, j = 1, ..., n-1$ with $|i-j| \geq 2$

(b) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $\quad i = 1, ..., n-2$



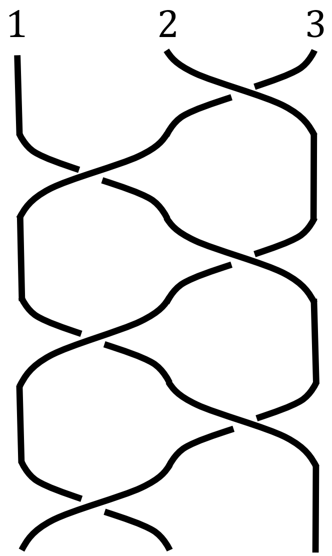
Action of σ_i



Action of σ_i^{-1}

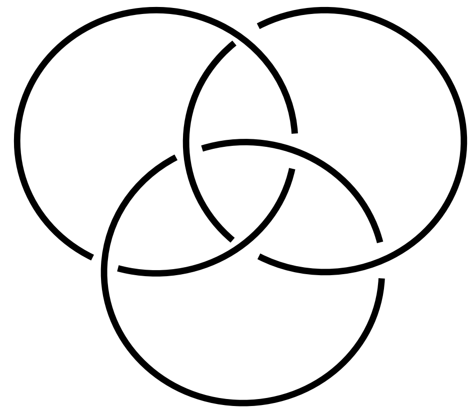
An **n-braid** $\sigma_i\sigma_j\sigma_k\cdots$, $(i, j, k, = 1, ..., n-1)$, is a set of n "strands" that carries a representation of B_n.

Claim. Every n-link can be represented by a closed n-braid.

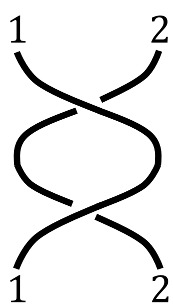


$\sigma_2{}^{-1}\sigma_1\sigma_2{}^{-1}\sigma_1\sigma_2{}^{-1}\sigma_1$

$1 \to 1,\ 2 \to 2,\ 3 \to 3$

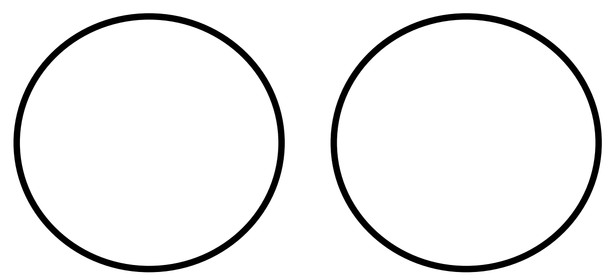


Borromean rings 3-link



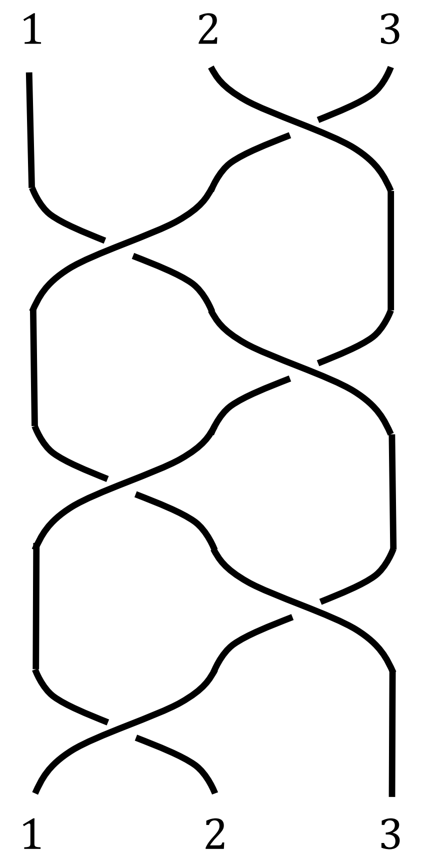
$\sigma_1\sigma_1{}^{-1}$

$1 \to 1,\ 2 \to 2$



trivial 2-link

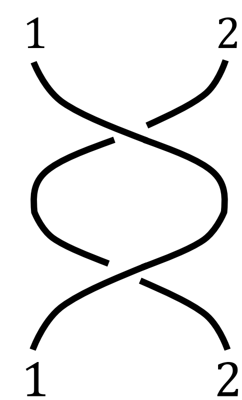
An n-braid $\sigma_i\,\sigma_j\sigma_k\cdots$ is **topologically entangled** just when it contains at least one pairwise set of terms $\sigma_i\sigma_j$ such that $\sigma_i\sigma_j \neq \mathbf{1}$.



$\sigma_2{}^{-1}\sigma_1\sigma_2{}^{-1}\sigma_1\sigma_2{}^{-1}\sigma_1$

Topologically entangled

Not topologically entangled



$\sigma_1\sigma_1{}^{-1}$

Def. 10 (*Topologically entangled n-link*). An n-link is **topologically entangled** just when it is the closure of a topologically entangled n-braid.

Def. 11 (*Quantum entangled vector state*). A state represented by a vector $|\psi\rangle \in \mathcal{H}$ is **quantum entangled** with respect to a decomposition $\mathcal{H} = V_1 \otimes \cdots \otimes V_n$ just when it cannot be expressed as a product of n terms, $|\psi\rangle = |v_1\rangle \otimes \cdots \otimes |v_n\rangle$, where $|v_i\rangle \in V_i$.

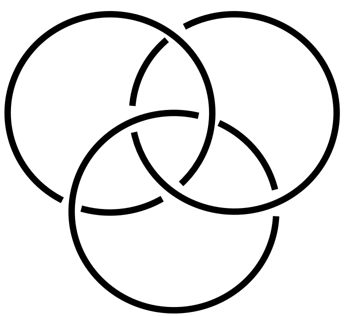
Are topologically entangled n-links the dual of quantum entangled n-partite states?

Alas, no...

(i) A topologically entangled n-link can correspond to many distinct quantum entangled n-partite states.

$$\alpha|\uparrow_{a_1}\rangle\{|\uparrow_{b_2}\rangle|\uparrow_{b_3}\rangle - |\downarrow_{b_2}\rangle|\downarrow_{b_3}\rangle\}$$
$$+ \beta|\downarrow_{a_1}\rangle\{|\uparrow_{b_2}\rangle|\uparrow_{b_3}\rangle - |\downarrow_{b_2}\rangle|\downarrow_{b_3}\rangle\}$$

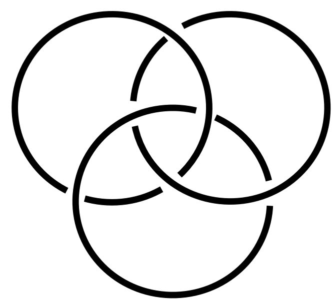
a, b are arbitrary spin axes



Three-Hopf rings

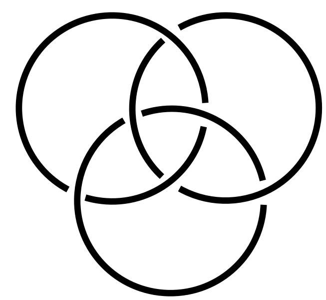
(ii) A quantum entangled n-partite state may not correspond to a definite topologically entangled n-link.

$$\sqrt{\frac{1}{3}}\,|\uparrow_{z_1}\rangle\{|\uparrow_{z_2}\rangle|\downarrow_{z_3}\rangle + |\downarrow_{z_2}\rangle|\uparrow_{z_3}\rangle\}$$
$$+ \sqrt{\frac{1}{3}}\,|\downarrow_{z_1}\rangle|\uparrow_{z_2}\rangle|\uparrow_{z_3}\rangle$$



Three-Hopf rings

or



Borromean rings

Observable-Based Approaches

Goal: Find suitable dual notions of "quantum entanglement observable" and "topological observable".

Kauffman & Lomonaco (2002)
Alagic, Jarret, & Jordan (2016)

A **unitary representation** $\tau_n^{(R)}$ of B_n is a map,

$$\tau_n^{(R)} \;:\; B_n \to V^{\otimes n}$$

$$\tau_n^{(R)}(\sigma_k) = I^{\otimes k-1} \otimes R \otimes I^{\otimes n-k-1}$$

where the Yang-Baxter (Y-B) operator $R : V \otimes V \to V \otimes V$ is defined by

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

Idea: Each n-braid b corresponds to an n-partite unitary operator $\tau_n^{(R)}(b)$ on $V^{\otimes n}$.

An n-partite operator O on $V^{\otimes n}$ is **quantum entangling** just when there is a product vector $|\Phi\rangle \in V^{\otimes n}$ such that $O|\Phi\rangle$ is a non-product vector.

Question 1: Is $\tau_n^{(R)}(b)$ quantum entangling if and only if b is topologically entangled?

Alas, no...

Kauffman & Lomonaco (2002)

$$\text{Let } R = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

(i) R is a Y-B operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(ii) R is not quantum entangling for $\alpha\beta = \delta\gamma$.

Claim. Let b be a topologically entangled 2-braid. Then $\tau_2{}^{(R)}(b)$ is not quantum entangling for $\alpha\beta = \delta\gamma$.

- *Not all that surprising...*
- *We already knew topologically entangled closed braids, as "topological states", are not dual to non-product vector states.*

How about link invariants as topological observables?

An **enhanced Y-B operator** $\mathcal{R}$ is a pair $\mathcal{R} = (R, \mu)$ where R is a Y-B operator and $\mu \in \mathrm{End}(V)$, such that

(i) $\mu \otimes \mu \cdot R = R \cdot \mu \otimes \mu$

(ii) $\mathrm{Tr}_2[R \cdot \mu \otimes \mu] = \mathrm{Tr}_2[R^{-1} \cdot \mu \otimes \mu] = \mu$

Claim. If $\mathcal{R}$ is an enhanced Y-B operator, then

$$I_{\mathcal{R}}(b) \equiv \mathrm{Tr}[\tau_n{}^{(R)}(b) \cdot \mu^{\otimes n}]$$

is an invariant of closed n-braids.

Question 2: *Is* $\tau_n{}^{(R)}(b)$ *quantum entangling if and only if* $I_{\mathcal{R}}(b)$ *is non-trivial?*

Alas, no...

Alagic, Jarret, & Jordan (2016)

Claim. Let $\mathcal{R}$ be an enhanced Y-B operator. If $\mathcal{R}$ is non-entangling, then $I_{\mathcal{R}}(b)$ is constant (hence trivial) on closed n-braids.

So: Quantum entangling of $\mathcal{R}$ is necessary (but not sufficient) for a non-trivial link invariant $I_{\mathcal{R}}(b)$.

- *Maybe we haven't identified the appropriate quantum entanglement observables...*
- *Maybe we haven't identified the appropriate topological entanglement observables...*

What is a quantum entanglement observable?

An observable that violates an entropic inequality?

For a bipartite state ρ_{AB}, an entropic inequality takes the general form:

$$S_\alpha(\rho_A) \leq S_\alpha(\rho_{AB})$$

where $S_\alpha(\rho) \equiv (1-\alpha)^{-1}\log\mathrm{Tr}\rho^\alpha$ is the α-Renyi entropy.

An observable that violates a Bell inequality?

For a bipartite state ρ_{AB} with respect to which the pairs of observables (O_A, O'_A), (O_B, O'_B) are conditionally statistically independent, the CHSH Bell inequality takes the form:

$$|\mathrm{Tr}(\mathcal{B}\rho_{AB})| \leq 2$$

where $\mathcal{B} = O_A \otimes (O_B + O'_B) + O'_A \otimes (O_B - O'_B)$.

A non-trivial distinction...

<u>*Example*</u>: Werner states

$$\rho_W = p|\Psi^-\rangle\langle\Psi^-| + (1-p)(I_2 \otimes I_2)/4$$

Claim.

(a) For $p > 1/3$, ρ_W is non-separable.

(b) For $p > 1/\sqrt{3} \approx 0.57735$, ρ_W violates the $\alpha = 2$ entropic inequality.

(c) For $p > 1/\sqrt{2} \approx 0.707107$, ρ_W violates the CHSH Bell inequality.

(d) For $p > 0.7476$, ρ_W violates the $\alpha = 1$ (von Neumann) entropic inequality.

Horodecki, *et al*. (1996)

Krammer (2005)

<u>*Thus*</u>:

- For $0.57735 < p < 0.707107$, the entanglement of ρ_W is detected by an entropic inequality but not a Bell inequality.
- For $0.70707 < p < 0.7476$, the entanglement of ρ_W is detected by a Bell inequality but not the von Neumann entropic inequality.

What is a topological entanglement observable?

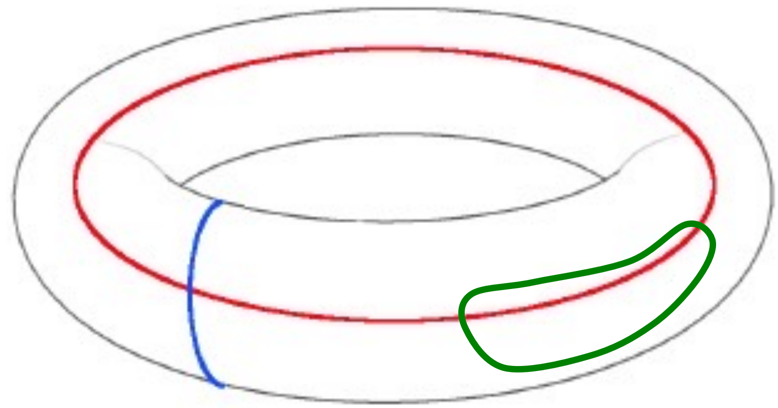
Issue #1: Non-Linearity. To the extent that quantum entanglement is characterized by an entropic inequality, it cannot be represented by a linear operator. But typical topological observables are linear.

- Non-linearity of quantum entanglement observables: Set of entangled states is not closed under addition!

- Linearity of topological observables:
  - *Wilson loop observables*
  - *Aharonov-Bohm phases*

Issue #2: Non-Locality. To the extent that quantum entanglement is characterized by a Bell inequality, it is linear. But it exhibits a non-locality that is distinct from topological non-locality.

Def. 12. An observable satisfies **localization** just when it has support at, or in a finite contractible neighborhood of, a point.



Def. 13. An observable exhibits **topological non-locality** just when it has support on a non-contractible region of spacetime.

Def. 14. Two observables exhibit **quantum entanglement non-locality** just when they exhibit a *distant, Bell inequality-violating correlation*.

- Topological non-locality violates localization.
- Quantum entanglement non-locality is consistent with localization.

Are there circumstances in which topological non-locality entails quantum entanglement non-locality?

Issue #3: Correlations. Can a topological observable enter into a Bell inequality-violating correlation?

4 Types of Correlation

(i) *Local* = involves only localizable observables.

(ii) *Non-local* = involves at least one non-localizable observable.

(iii) *Short-range* = goes to zero at sufficiently large separation distances.

(iv) *Long-range* = does not go to zero at sufficiently large separation distances.

- Quantum entanglement correlations (distant, Bell inequality-violating) are typically viewed as type (i), and can be (iii) or (iv).
- Type (ii) seems relevant to case of topologically non-local observables.
- Can type (ii) underwrite a duality between topological and quantum entanglement?
  - *Can type (ii) be explained by direct causes and/or common causes?*

Conclusion

- RT Formula as an example of a duality between spacetime topology and quantum entanglement.
- Two approaches to understanding such a duality.
  - *State-based: n-links are dual to n-partite states.*
  - *Observable-based: certain topological observables are dual to entanglement observables.*
- Both approaches have their issues!

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