Spacetime Structuralism

Jonathan Bain
Polytechnic University

Overview:

If it is possible to do classical field theory without a 4-dimensional differentiable manifold, what does this suggest about the ontological status of spacetime?

I. *Why* would we want to do classical field theory without a manifold?

II. The extent to which such a feat is possible:

1. Twistors
2. Einstein algebras
3. Geometric algebra

III. What type of spacetime realism does this feat suggest?

I. Why would we want to do classical field theory without a manifold?

Tensor formalism: Classical field theory given by

$$(M,\ \partial_a,\ O_i)$$

Spacetime represented by 4-dim differentiable manifold.

... formulated in terms of a derivative operator.

Physical fields represented by tensor fields defined on M that satisfy appropriate field equations...

Ex. CED in Minkowski spacetime

$$(M,\ \eta_{ab},\ \partial_a,\ F_{ab},\ J^a)$$

$$\left.\begin{aligned} \partial_a F^{ab} &= 4\pi J^b \\ \partial_{[a} F_{ab]} &= 0 \end{aligned}\right\} \text{Maxwell's Equations}$$

The Roles Played by M:

(1) *Kinematical*: As the support structure on which tensor fields are defined. In this role, *M* provides the mathematical wherewithal for representations of physical fields to be defined.

(2) *Dynamical*: As the support structure on which derivative operators are defined. In this role, *M* provides the mathematical wherewithal for a dynamical description of the evolution of physical fields in the form of field equations.

Manifold Substantivalism: Ontological commitment to spacetime points

(i) Manifold points represent real spacetime points. (*Substantivalism*)

(ii) Diffeomorphically related models of classical field theories in the tensor formalism represent distinct physically possible worlds. (*Denial of Leibniz Equivalence*)

BOTH (i) and (ii) are motivated by a desire to *literally interpret* classical field theories.

$\Longleftrightarrow$

Semantic realism: Desires to take successful theories at their face value.

***BUT*:** **Hole Argument:** (i) & (ii) $\Rightarrow$ *Indeterminism!*

Can spacetime realism be better motivated in formalisms in which M does not appear?

II. Doing Away with M, Part 1: Twistors

$$\begin{tikzcd} & \boldsymbol{F} \arrow[dl, "\mu"'] \arrow[dr, "\upsilon"] & \\ \boldsymbol{PT} & & \boldsymbol{CM^c} \end{tikzcd}$$

Projective twistor space | *Complex, compactified Minkowski spacetime*

Projection maps

$\upsilon : (x^a, \pi_{A'}) \to x^a$

$\mu : (x^a, \pi_{A'}) \to (ix^{AA'}\pi_{A'}, \pi_{A'})$

T = carrier space for $SU(2, 2)$ = double covering group of $SO(2, 4)$ = double covering group of $C(1, 3)$!

$SU(2, 2) \xrightarrow{\text{2-1}} SO(2, 4) \xrightarrow{\text{2-1}} C(1, 3)$

M^c = carrier space for 4-dim conformal group $C(1, 3)$ of conformal transformations on Minkowski spacetime.

Geometric Interpretation of PT:

Klein Correspondence

$\omega^A = ix^{AA'}\pi_{A'}$

PT	CM^c
point	α-plane
line	point
line in PN	real point
point in PN	real null geodesic
point in $PT^+ \cup PT^-$	real Robinson congruence
intersection of lines	null separation of points

complex null-plane

twisted null congruence

"Stone-Age" Twistor Theory

- To do classical field theory using twistors, need to identify those field-theoretic structures on CM^c that can be pulled up to F and then pushed down to PT. Major results:

Fields and derivative operators in CM^c		*Purely geometrical structures on PT*
Spinor fields o^A that satisfy the null shear-free geodesic equation: $o^A o^B \partial_{BB'} o_A = 0$	$\Longleftrightarrow$ *Kerr's theorem*	Intersection of PN with surface Q defined by $f(Z^\alpha) = 0$ for some function f on PT.
Zero rest-mass fields $\varphi_{A'...B'}(x)$, $\varphi_{A'...B'}(x)$: $\partial^{AA'}\phi_{A'...B'}(x) = 0, \quad \partial^{AA'}\varphi_{A...B}(x) = 0$	$\Longleftrightarrow$ *Zero rest-mass Penrose transformation*	Cohomology groups on PT: $H^1(PT^+; \mathcal{O}(-n-2))$, $H^1(PT^-; \mathcal{O}(n-2))$
Anti-self-dual Yang-Mills fields F_{ab}: $\partial_a F_{bc} = 0, \quad \partial_{[a}F_{bc]} = 0, \quad {}^*F_{ab} = -iF_{ab}$	$\Longleftrightarrow$ *Ward's theorem*	Vector bundle over PT trivial on twistor lines.

Additional twistor constructions for

- Vacuum solutions to the Einstein equations with anti-self-dual Weyl curvature (*Non-Linear Graviton*).
- Stationary axi-symmetric vacuum solutions to the Einstein equations.
- Extensions of *ZRMPT* for fields with sources.
- Extensions of *Ward's theorem* for other non-linear integrable field equations.
- Real analytic vacuum Einstein spacetimes in general (Sparling 1998).

Interpretation

In what sense does the twistor formalism do away with M?

(a) *Dynamical role* of *M*: Unnecessary in twistor formalism -- evaporation of local field equations into global geometric/holomorphic structure.

(b) *Kinematical role* of *M*: Played by appropriate geometrical structures over *PT*.

Tensor formalism

Tensor fields quantifying over manifold points

Individuals-based ontology:

(1) Physical fields

(2) Objects of predication (spacetime points, loops, *etc*)

Qualifications:

(a) Which fields? (gauge potential A_a or gauge field F_{ab})

(b) Which spacetime objects? (points or loops)

Twistor formalism

Geometrical structures quantifying over projective twistors

Individuals-based ontology:

Twistors

(1) Twisted null geodesics

(2) States of zero-rest-mass particles

(3) Charges for spin-3/2 fields

(4) Edge states of a 4-dim fermionic Quantum Hall Effect fluid (Sparling 2002)

II. Doing Away with M, Part 2: Einstein Algebras

point set → topology → maximal atlas → differentiable manifold

commutative ring → differentiable structure → differentiable manifold

$C^\infty(M)$ = commutative ring of real-valued smooth functions on M

$C^c(M)$ = subring of constant real-valued smooth functions on M

Derivation on $(C^\infty(M), C^c(M))$ = a map $X : C^\infty(M) \to C^\infty(M)$, such that

$X(f + g) = Xf + Xg, \quad X(fg) = fX(g) + X(f)g, \quad X(f) = 0$, for $f \in C^c(M)$

[The space of derivations on $(C^\infty(M), C^c(M))$ is isomorphic to the space of smooth contravariant vector fields on M.] ⇒ [All tensorial objects defined on M can be reconstructed from $C^\infty(M)$]

Einstein Algebra (EA) formalism:

Classical field theory given by:

$$(\mathcal{R}^\infty, \mathcal{R}, A_i)$$

Called an Einstein algebra for restriction to *GR* (Geroch 1972)

commutative ring

subring of $\mathcal{R}^\infty$ isomorphic to $\mathbb{R}$

multilinear maps on space of derivations of $(\mathcal{R}^\infty, \mathcal{R})$ and its dual, and satisfying appropriate field equations

Extensions of Einstein Algebras

	Tensor formalism	*EA formalism*	
Non-singular spacetimes	M	$C^{\infty}(M)$	*Geroch (1972)*
Singular spacetimes	$M' = M \cup \partial_b M$	$C^{\infty}(M')$ = sheaf of Einstein algebras	*Heller & Sasin (1995)*
		OR	
		non-commutative Einstein algebra of $\mathbb{C}$-valued smooth functions on $OM \times O(1, 3)$	*Heller & Sasin (1996)*
Schema for Quantum Gravity		$\mathcal{E}$ = $\mathcal{C}$*-Einstein algebra = non-commutative Einstein algebra of $\mathbb{C}$-valued smooth functions on a groupoid	*Heller & Sasin (1999)*

Interpretation

In what sense does the EA formalism do away with M?

1. *Trivial sense*: *M done away with only in name.*

   The points of M can be reconstructed from the maximal ideals of $C^\infty(M)$.

$$(M, \partial_a, O_i) \xrightarrow{\text{1-1}} (\mathcal{R}^\infty, \mathcal{R}, A_i)$$

   Consequence: Nothing gained by *EA*? Any interpretive options under consideration in tensor formalism will translate 1-1 into *EA* formalism.

2. *Non-trivial sense: M really done away with.*

   Evident in extensions of *EA* formalism to non-singular spacetimes. Commutative algebras replaced by non-commutative algebras which, in general, have no maximal ideals.

Abstract Einstein algebra $\mathcal{A}$

Realizable on variety of spaces:

- differentiable manifold
- differentiable manifold w/boundary
- geometric structures over manifolds
- groupoids

Suggests literal interpretation:

- not of any concrete representation of $\mathcal{A}$
- but of structure defined by algebraic properties of $\mathcal{A}$

II. Doing Away with M, Part 3: Geometric Algebra

Geometric algebra $\mathcal{G}$ = generalization of a vector space.

1. Start with n-dim vector space V^n.
2. Equip V^n with $\mathbb{R}$-valued inner product, $\bullet\ :\ V^n \times V^n \to \mathbb{R}$.
3. Equip V^n with wedge product, $\wedge\ :\ \Lambda^p V^n \times \Lambda^q V^n \to \Lambda^{p+q} V^n$.
4. Define *geometric product* on V^n as $ab \equiv a \bullet b + a \wedge b$, for $a, b \in V^n$.
5. Now form real associative algebra $\mathcal{G}(V^n)$ closed under addition and geometric product.
6. $\mathcal{G}(V^n)$ is *graded*. Decomposable as:

$$\underset{\substack{\mathbb{R}\\ \textit{scalars}}}{\mathcal{G}^0} + \underset{\substack{V^n\\ \textit{vectors}}}{\mathcal{G}^1} + \underset{\textit{bivectors}}{\mathcal{G}^2} + \ldots + \underset{\textit{pseudoscalar}}{\mathcal{G}^n}$$

7. For V^n equipped with bilinear form with signature (p, q), $\mathcal{G}(V^n)$ is the real Clifford algebra $\mathcal{C}_{(p,q)}$.

The *Dirac algebra* $\mathcal{D} = \mathcal{G}(M^4)$, where M^4 = Minkowski vector space.

- Generated by the set of 1-vectors $\{\gamma_\mu\}$, $\mu = 0...3$, satisfying $\gamma_0\gamma_0 = 1$, $\gamma_k\gamma_k = -1$ and $\gamma_\mu \bullet \gamma_\upsilon = 0$ for $\mu \neq \upsilon$.
- Basis given by:

$$\underbrace{\{1}_{\mathcal{D}^0 \cong \mathbb{R}},\ \underbrace{\{\gamma_\mu\}}_{\mathcal{D}^1 \cong M^4},\ \underbrace{\{\sigma_k, i\sigma_k\}}_{\mathcal{D}^2},\ \underbrace{\{i\gamma_\mu\}}_{\mathcal{D}^3},\ \underbrace{i\}}_{\mathcal{D}^4}$$

Minkowski metric recovered as $\eta_{\mu\nu} = \gamma_\mu \bullet \gamma_\upsilon$

where the pseudoscalar is given by $i \equiv \gamma_0\gamma_1\gamma_2\gamma_3$, and $\sigma_k \equiv \gamma_k\gamma_0$, $k = 1...3$, are bivectors that form an orthonormal frame in the Euclidean 3-space orthogonal to the γ_0 direction.

Classical field theories in Minkowski spacetime:

$$(\mathcal{D}, \partial, O_i)$$

Dirac algebra replaces M

... formulated in terms of a vector derivative operator defined on $\mathcal{D}$.

Physical fields represented by geometric functions defined on $\mathcal{D}$ that satisfy appropriate field equations...

<u>Ex. CED in Minkowski spacetime</u>

$$(\mathcal{D}, \partial, F, J)$$

bivector-valued function on $\mathcal{D}^1$

vector-valued function on $\mathcal{D}^1$

$$\partial F = 4\pi J \quad \begin{cases} \partial \bullet F = 4\pi J \\ \partial \wedge F = 0 \end{cases}$$

Maxwell's Equations

Interpretation

In what sense does the GA formalism do away with M?

Kinematical and dynamical roles of M are played by the Dirac algebra in its entirety.

$$\mathcal{D} = \mathcal{D}^0\ (\cong \mathbb{R}) + \mathcal{D}^1\ (\cong M^4) + \mathcal{D}^2 + \mathcal{D}^3 + \mathcal{D}^4$$

$\mathcal{D}$ comes pre-packaged with

(a) derivative operators
(b) representations of physical fields (objects of relevant subalgebras)
(c) metrical structure of M^4

Literal Interpretations: Options

(i) [*Intended interpretation*: Elements of $\mathcal{D}$ = multivectors = fundamental geometrical objects] ⇒ [*Relationalism*: Spacetime arising out of relations between multivectors in $\mathcal{D}$.]

(ii) [*Structural interpretation*: Nature of elements of $\mathcal{D}$ left unspecified...] ⇒ [Spacetime consists of structure defined by algebraic properties of $\mathcal{D}$ (*i.e.*, $\mathcal{C}_{(1,\,3)}$).]

III. Spacetime Structuralism

Fundamentalism is in the eye of the beholder...

Tensor formalism:
Point set fundamental.
Differentiable, conformal, metrical structures derivative.

Twistor formalism:
Conformal structure fundamental.
Point set, differentiable, metrical structures are derivative.

EA formalism:
Differentiable structure fundamental.
Point set, conformal, metrical structures derivative.

GA formalism:
Metrical structure fundamental.
Point set, differentiable, conformal structures derivative.

A debate over structure minimally required to support mathematical representations of physical fields.

Not a debate over how this structure manifests itself:

- What it is predicated on.
- The nature of the mathematical objects used to describe it.

Semantic realism with respect to classical field theories... $\Longrightarrow$ Ontological commitment to structure that is minimally required to support representations of fields.

Spacetime Realism as Spacetime Structuralism

(a) *Not* substantivalism (*i.e.*, not a commitment to spacetime points).

(b) *Not* relationalism (*i.e.*, not anti-realism with respect to spacetime).

(c) A commitment to spacetime structure.