

Spacetime Structuralism

Jonathan Bain
Polytechnic University

Overview:

If it is possible to do classical field theory without a 4-dimensional differentiable manifold, what does this suggest about the ontological status of spacetime?

I. *Why* would we want to do classical field theory without a manifold?

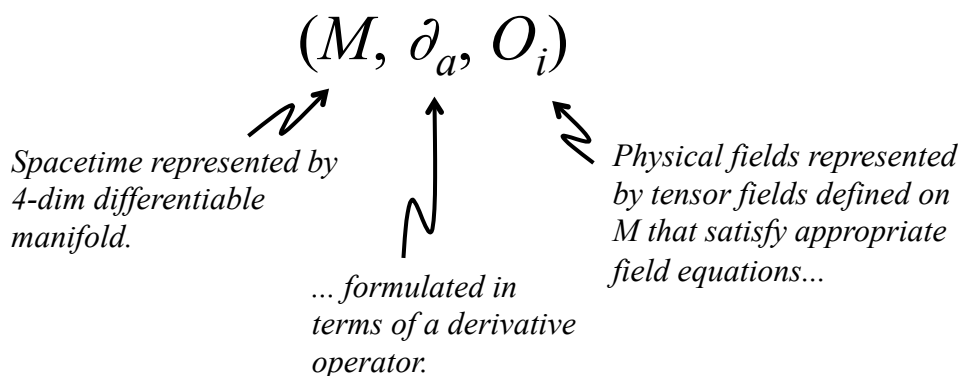
II. The extent to which such a feat is possible:

1. Twistors
2. Einstein algebras
3. Geometric algebra

III. What type of spacetime realism does this feat suggest?

I. Why would we want to do classical field theory without a manifold?

Tensor formalism: Classical field theory given by



Ex. CED in Minkowski spacetime

$$(M, \eta_{ab}, \partial_a, F_{ab}, J^a)$$

$$\partial_a F^{ab} = 4\pi J^b$$

$$\partial_{[a} F_{ab]} = 0$$

Maxwell's Equations

The Roles Played by M :

(1) Kinematical: As the support structure on which tensor fields are defined.

In this role, M provides the mathematical wherewithal for representations of physical fields to be defined.

(2) Dynamical: As the support structure on which derivative operators are defined.

In this role, M provides the mathematical wherewithal for a dynamical description of the evolution of physical fields in the form of field equations.

Manifold Substantivalism: Ontological commitment to spacetime points

- (i) Manifold points represent real spacetime points. (*Substantivalism*)
- (ii) Diffeomorphically related models of classical field theories in the tensor formalism represent distinct physically possible worlds. (*Denial of Leibniz Equivalence*)

BOTH (i) and (ii) are motivated by a desire to literally interpret classical field theories.

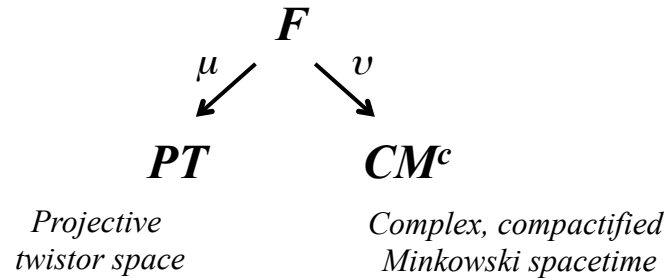


Semantic realism: Desires to take successful theories at their face value.

BUT: Hole Argument: (i) & (ii) \Rightarrow *Indeterminism!*

Can spacetime realism be better motivated in formalisms in which M does not appear?

II. Doing Away with M , Part 1: Twistors



Projection maps

$$v : (x^a, \pi_{A'}) \rightarrow x^a$$

$$\mu : (x^a, \pi_{A'}) \rightarrow (ix^{AA'} \pi_{A'}, \pi_{A'})$$

T = carrier space for $SU(2, 2)$ =
 double covering group of $SO(2, 4)$ =
 double covering group of $C(1, 3)$!

$$SU(2, 2) \xrightarrow{2-1} SO(2, 4) \xrightarrow{2-1} C(1, 3)$$

M^c = carrier space for 4-dim
 conformal group $C(1, 3)$ of
 conformal transformations on
 Minkowski spacetime.

Geometric Interpretation of PT :

Klein Correspondence

$$\omega^A = ix^{AA'} \pi_{A'}$$

PT	CM^c	
point	α -plane	← <i>complex null-plane</i>
line	point	
line in PN	real point	
point in PN	real null geodesic	
point in $PT^+ \cup PT^-$	real Robinson congruence	← <i>twisted null congruence</i>
intersection of lines	null separation of points	

- To do classical field theory using twistors, need to identify those field-theoretic structures on CM^c that can be pulled up to F and then pushed down to PT . Major results:

Fields and derivative operators in CM^c

Spinor fields σ^A that satisfy the null shear-free geodesic equation:

$$\sigma^A \sigma^B \partial_{BB'} \sigma_A = 0$$

\longleftrightarrow
Kerr's theorem

Purely geometrical structures on PT

Intersection of PN with surface Q defined by $f(Z^\alpha) = 0$ for some function f on PT .

Zero rest-mass fields $\varphi_{A'...B'}(x)$, $\varphi_{A...B}(x)$:

$$\partial^{AA'} \varphi_{A'...B'}(x) = 0, \quad \partial^{AA'} \varphi_{A...B}(x) = 0$$

\longleftrightarrow
Zero rest-mass Penrose transformation

Cohomology groups on PT :
 $H^1(PT^+; \mathcal{O}(-n-2))$,
 $H^1(PT^-; \mathcal{O}(n-2))$

Anti-self-dual Yang-Mills fields F_{ab} :

$$\partial_a F_{bc} = 0, \quad \partial_{[a} F_{bc]} = 0, \quad *F_{ab} = -iF_{ab}$$

\longleftrightarrow
Ward's theorem

Vector bundle over PT trivial on twistor lines.

Additional twistor constructions for

- Vacuum solutions to the Einstein equations with anti-self-dual Weyl curvature (*Non-Linear Graviton*).
- Stationary axi-symmetric vacuum solutions to the Einstein equations.
- Extensions of *ZRMPT* for fields with sources.
- Extensions of *Ward's theorem* for other non-linear integrable field equations.
- Real analytic vacuum Einstein spacetimes in general (Sparling 1998).

Interpretation

In what sense does the twistor formalism do away with M ?

- (a) *Dynamical role of M* : Unnecessary in twistor formalism -- evaporation of local field equations into global geometric/holomorphic structure.
- (b) *Kinematical role of M* : Played by appropriate geometrical structures over PT .

Tensor formalism

Tensor fields quantifying over manifold points

Individuals-based ontology:

- (1) Physical fields
- (2) Objects of predication (spacetime points, loops, *etc*)

Qualifications:

- (a) Which fields? (gauge potential A_a or gauge field F_{ab})
- (b) Which spacetime objects? (points or loops)

Twistor formalism

Geometrical structures quantifying over projective twistors

Individuals-based ontology:

Twistors

- (1) Twisted null geodesics
- (2) States of zero-rest-mass particles
- (3) Charges for spin-3/2 fields
- (4) Edge states of a 4-dim fermionic Quantum Hall Effect fluid
(Sparling 2002)

II. Doing Away with M, Part 2: Einstein Algebras

point set \rightarrow topology \rightarrow maximal atlas \rightarrow differentiable manifold

commutative ring \rightarrow differentiable structure \rightarrow differentiable manifold



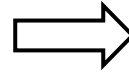
$C^\infty(M)$ = commutative ring of real-valued smooth functions on M

$C^c(M)$ = subring of constant real-valued smooth functions on M

Derivation on $(C^\infty(M), C^c(M))$ = a map $X : C^\infty(M) \rightarrow C^\infty(M)$, such that

$X(f+g) = Xf + Xg$, $X(fg) = fX(g) + X(f)g$, $X(f) = 0$, for $f \in C^c(M)$

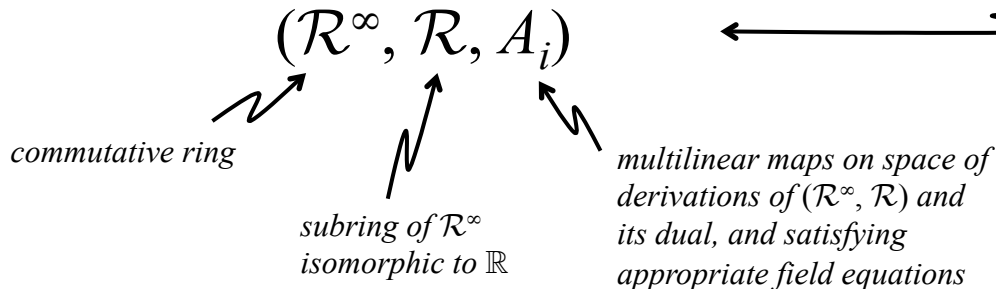
$\left[\begin{array}{l} \text{The space of derivations on } (C^\infty(M), C^c(M)) \\ \text{is isomorphic to the space of smooth} \\ \text{contravariant vector fields on } M. \end{array} \right]$



$\left[\begin{array}{l} \text{All tensorial objects} \\ \text{defined on } M \text{ can be} \\ \text{reconstructed from } C^\infty(M) \end{array} \right]$

Einstein Algebra (EA) formalism:

Classical field theory given by:



\longleftarrow Called an Einstein algebra for restriction to GR (Geroch 1972)

Extensions of Einstein Algebras

	<i>Tensor formalism</i>	<i>EA formalism</i>	
<i>Non-singular spacetimes</i>	M	$C^\infty(M)$	<i>Geroch (1972)</i>
<i>Singular spacetimes</i>	$M' = M \cup \partial_b M$	$C^\infty(M')$ = sheaf of Einstein algebras OR non-commutative Einstein algebra of \mathbb{C} -valued smooth functions on $OM \times O(1, 3)$	<i>Heller & Sasin (1995)</i> <i>Heller & Sasin (1996)</i>
<i>Schema for Quantum Gravity</i>		\mathcal{E} = C^* -Einstein algebra = non-commutative Einstein algebra of \mathbb{C} -valued smooth functions on a groupoid	<i>Heller & Sasin (1999)</i>

Interpretation

In what sense does the EA formalism do away with M ?

1. *Trivial sense: M done away with only in name.*

The points of M can be reconstructed from the maximal ideals of $C^\infty(M)$.

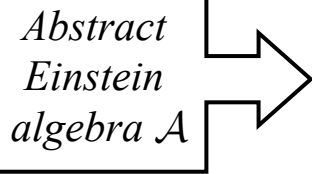
$$(M, \partial_a, O_i) \xrightarrow{1-1} (\mathcal{R}^\infty, \mathcal{R}, A_i)$$

Consequence: Nothing gained by EA? Any interpretive options under consideration in tensor formalism will translate 1-1 into EA formalism.

2. *Non-trivial sense: M really done away with.*

Evident in extensions of EA formalism to non-singular spacetimes. Commutative algebras replaced by non-commutative algebras which, in general, have no maximal ideals.

Abstract
Einstein
algebra \mathcal{A}



Realizable on variety of spaces:

- differentiable manifold
- differentiable manifold w/boundary
- geometric structures over manifolds
- groupoids

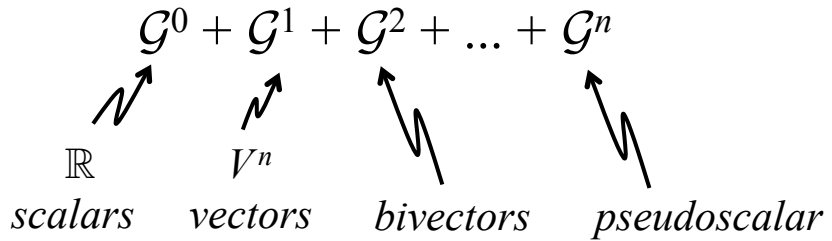
Suggests literal interpretation:

- not of any concrete representation of \mathcal{A}
- but of structure defined by algebraic properties of \mathcal{A}

II. Doing Away with M, Part 3: Geometric Algebra

Geometric algebra \mathcal{G} = generalization of a vector space.

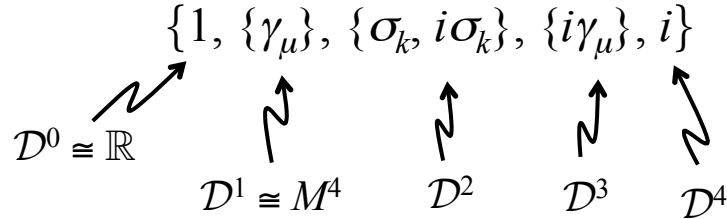
1. Start with n -dim vector space V^n .
2. Equip V^n with \mathbb{R} -valued inner product, $\bullet : V^n \times V^n \rightarrow \mathbb{R}$.
3. Equip V^n with wedge product, $\wedge : \Lambda^p V^n \times \Lambda^q V^n \rightarrow \Lambda^{p+q} V^n$.
4. Define *geometric product* on V^n as $ab \equiv a \bullet b + a \wedge b$, for $a, b \in V^n$.
5. Now form real associative algebra $\mathcal{G}(V^n)$ closed under addition and geometric product.
6. $\mathcal{G}(V^n)$ is *graded*. Decomposable as:



7. For V^n equipped with bilinear form with signature (p, q) , $\mathcal{G}(V^n)$ is the real Clifford algebra $\mathcal{C}_{(p, q)}$.

The Dirac algebra $\mathcal{D} = \mathcal{G}(M^4)$, where $M^4 =$ Minkowski vector space.

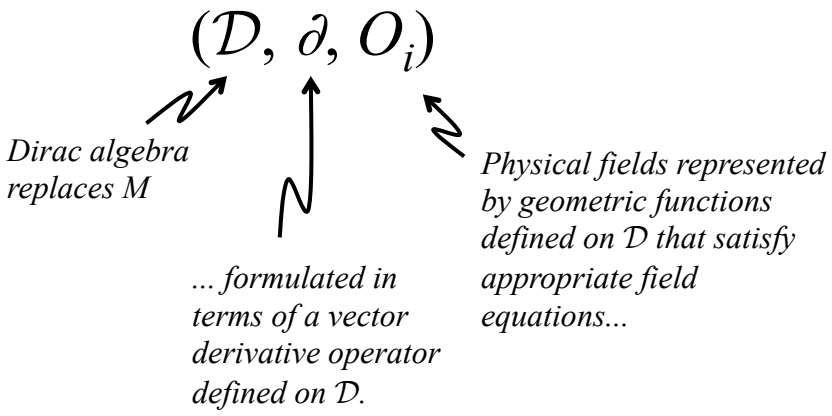
- Generated by the set of 1-vectors $\{\gamma_\mu\}$, $\mu = 0...3$, satisfying $\gamma_0\gamma_0 = 1$, $\gamma_k\gamma_k = -1$ and $\gamma_\mu \cdot \gamma_\nu = 0$ for $\mu \neq \nu$.
- Basis given by:



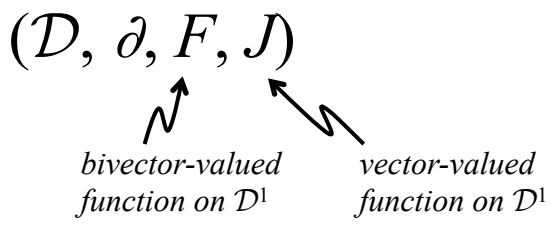
Minkowski metric recovered as $\eta_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu$

where the pseudoscalar is given by $i \equiv \gamma_0\gamma_1\gamma_2\gamma_3$, and $\sigma_k \equiv \gamma_k\gamma_0$, $k = 1...3$, are bivectors that form an orthonormal frame in the Euclidean 3-space orthogonal to the γ_0 direction.

Classical field theories in Minkowski spacetime:



Ex. CED in Minkowski spacetime



$$\partial F = 4\pi J \quad \left\{ \begin{array}{l} \partial \cdot F = 4\pi J \\ \partial \wedge F = 0 \end{array} \right. \quad \text{Maxwell's Equations}$$

Interpretation

In what sense does the GA formalism do away with M ?

Kinematical and dynamical roles of M are played by the Dirac algebra in its entirety.

$$\mathcal{D} = \mathcal{D}^0 (\cong \mathbb{R}) + \mathcal{D}^1 (\cong M^4) + \mathcal{D}^2 + \mathcal{D}^3 + \mathcal{D}^4$$

\mathcal{D} comes pre-packaged with

- (a) derivative operators
- (b) representations of physical fields (objects of relevant subalgebras)
- (c) metrical structure of M^4

Literal Interpretations: Options

(i) $\left[\begin{array}{l} \textit{Intended interpretation:} \\ \text{Elements of } \mathcal{D} = \text{multivectors} = \\ \text{fundamental geometrical objects} \end{array} \right] \Rightarrow \left[\begin{array}{l} \textit{Relationalism:} \\ \text{Spacetime arising out of relations} \\ \text{between multivectors in } \mathcal{D}. \end{array} \right]$

(ii) $\left[\begin{array}{l} \textit{Structural interpretation:} \\ \text{Nature of elements of } \mathcal{D} \text{ left} \\ \text{unspecified...} \end{array} \right] \Rightarrow \left[\begin{array}{l} \text{Spacetime consists of} \\ \text{structure defined by algebraic} \\ \text{properties of } \mathcal{D} \text{ (i.e., } \mathcal{C}_{(1,3)} \text{).} \end{array} \right]$

III. Spacetime Structuralism

Fundamentalism is in the eye of the beholder...

Tensor formalism:

Point set fundamental.

Differentiable, conformal, metrical structures derivative.

Twistor formalism:

Conformal structure fundamental.

Point set, differentiable, metrical structures are derivative.

EA formalism:

Differentiable structure fundamental.

Point set, conformal, metrical structures derivative.

GA formalism:

Metrical structure fundamental.

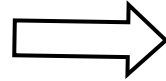
Point set, differentiable, conformal structures derivative.

A debate over structure minimally required to support mathematical representations of physical fields.

Not a debate over how this structure manifests itself:

- What it is predicated on.
- The nature of the mathematical objects used to describe it.

Semantic realism with respect to classical field theories...



Ontological commitment to structure that is minimally required to support representations of fields.

Spacetime Realism as Spacetime Structuralism

- (a) *Not* substantivalism (*i.e.*, not a commitment to spacetime points).
- (b) *Not* relationalism (*i.e.*, not anti-realism with respect to spacetime).
- (c) A commitment to spacetime structure.