QFTs in Classical Spacetimes and Particles

Jonathan Bain

Dept. of Humanities and Social Sciences
Polytechnic Institute of NYU
Brooklyn, New York

1. The Received View
2. Minkowski *vs.* Classical Spacetimes
3. Axioms for QFTs and Implications
4. Conclusion

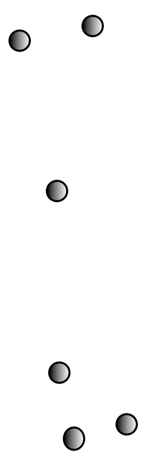
1. The Received View

What is a particle?

(1) Something _localizable_...

Here, **now**. _Not there, **now**._

(2) Something _countable_...



Two here, **now**.

$+$

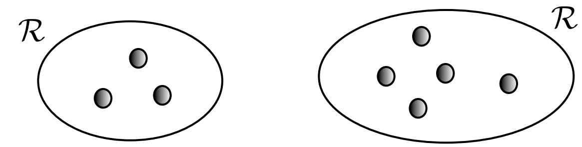
One here, **now**.

$=$

Three here, **now**.

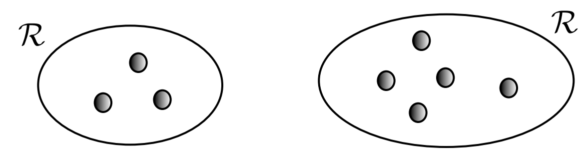
Mathematical translation:

(1) A Fock space formulation of the theory exists that admits _local number operators_.



$$N_{\mathcal{R}}|state\rangle = 3|state\rangle$$
$$N_{\mathcal{R}'}|state\rangle = 5|state\rangle$$

(2) A _unique_ Fock space formulation of the theory exists that admits a _total number operator_.



$$N = \int N_{\mathcal{R}} d^3\mathbf{x}$$

$$N|state\rangle = 8|state\rangle$$

1. The Received View

Claim: Relativistic Quantum Field Theories (RQFTs) do not admit particle interpretations.

Why?

- ***No*** local number operators in RQFTs (Reeh-Schlieder theorem).
- ***No*** unique total number operator in non-interacting RQFTs (Unruh Effect).
- ***No*** total number operator in interacting RQFTs (Haag's theorem).

Against the Received View:

The existence of an absolute temporal metric is a necessary condition for the existence of local number operators, and a unique total number operator.

Moral:

The Received View's concept of particle is informed by *non-relativistic* notions of localizability and countability associated with absolute concepts of space and time.

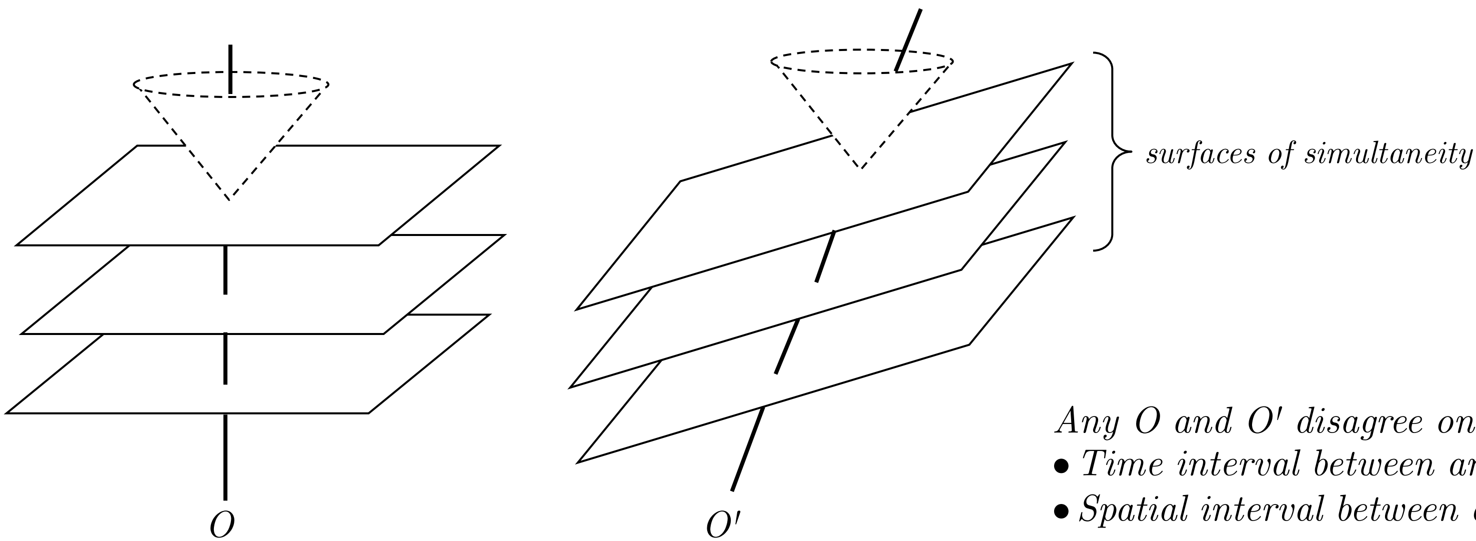
2. Minkowski *vs.* Classical Spacetimes

- ***Arena for Relativistic QFTs***: Minkowski spacetime $(M, \boldsymbol{\eta}_{ab})$.

$$\nabla_c \boldsymbol{\eta}_{ab} = 0$$

- $\boldsymbol{\eta}_{ab}$ determines a unique curvature tensor, which vanishes: Minkowski spacetime is *spatiotemporally flat*.

- No unique way to separate time from space:



Any O and O' disagree on:
- *Time interval between any two events.*
- *Spatial interval between any two events.*

- Symmetry group generated by $\pounds_x \boldsymbol{\eta}_{ab} = 0$. (*Poincaré group*)

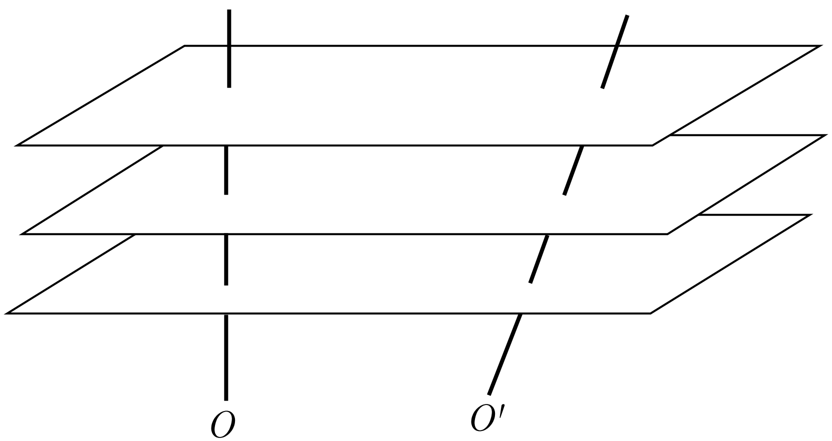
2. Minkowski *vs.* Classical Spacetimes

- ***Arena for Non-relativistic QFTs***: Classical spacetimes $(M,\ h^{ab},\ t_a,\ \nabla_a)$.

$$h^{ab}t_b = 0, \qquad \nabla_c h^{ab} = 0 = \nabla_a t_b$$

- h^{ab}, t_a fail to determine a unique curvature tensor! Allows additional constraints on curvature that define different types of classical spacetimes.

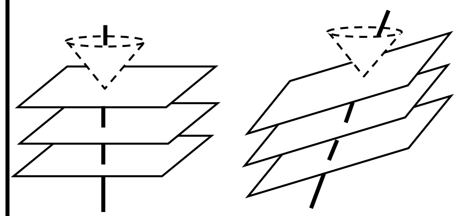
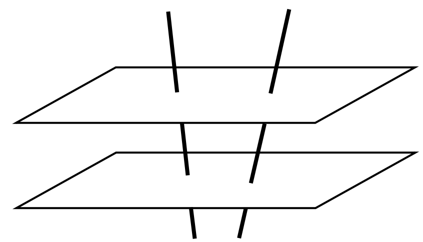
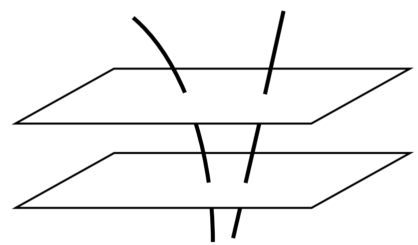
- Unique way exists to separate time from space:



Any O and O' agree on:
- *Time interval between any two events.*
- *Spatial interval between any two simultaneous events.*

- Symmetry group generated by $\pounds_x h^{ab} = \pounds_x t_a = 0$.

2. Minkowski *vs.* Classical Spacetimes

Spacetime	Absolute quantities	Relative quantities	Indistinguishable trajectories	Symmetries
<u>*Minkowski*</u> $\nabla_c \eta_{ab} = 0$ $(R^a{}_{bcd} = 0)$ *(spacetime flatness)*	• rotation • acceleration	• temporal intervals • spatial intervals • velocity		*(Poincaré group)* $x^\mu \mapsto x^{\mu\prime} = \Lambda^\mu{}_\nu \, x^\nu + d^\mu$
<u>*Neo-Newtonian*</u> $h^{ab} t_b = 0$ $\nabla_c h^{ab} = 0 = \nabla_a t_b$ $R^a{}_{bcd} = 0$ *(spacetime flatness)*	• temporal intervals • spatial intervals btw sim events • rotation • acceleration	• velocity		*(Galilei group)* $\mathbf{x} \mapsto \mathbf{x}' = R\mathbf{x} + \mathbf{v}t + \mathbf{a}$ $t \mapsto t' = t + b$
<u>*Maxwellian*</u> $h^{ab} t_b = 0$ $\nabla_c h^{ab} = 0 = \nabla_a t_b$ $R^{ab}{}_{cd} = 0$ *(rotation standard)*	• temporal intervals • spatial intervals btw sim events • rotation	• accel. • velocity		*(Maxwell group)* $\mathbf{x} \mapsto \mathbf{x}' = R\mathbf{x} + \mathbf{c}(t)$ $t \mapsto t' = t + b$

2. Minkowski *vs.* Classical Spacetimes

- **Relativistic quantum field theory (RQFT)** = A QFT invariant under *Poincaré*.

- **Non-relativistic quantum field theory (NQFT)** = A QFT invariant under a symmetry group that contains the classical spacetime symmetry group as a subgroup.

  - **Galilei-invariant QFT (GQFT)** = An NQFT invariant under *Gal*. (Lévy-Leblond 1967).

  - **Maxwell-invariant QFT (MQFT)** = An NQFT invariant under *Max*. (Christian 1997).

3. Axioms for QFTs

RQFT Axioms (Wightman 1956)

(W1) ***Fields***. The fundamental dynamical variables are local field operators that act on a Hilbert space $\mathcal{H}$ of states.

(W2) ***Poincaré Symmetry***. $\mathcal{H}$ admits a unitary projective representation of the Poincaré group.

(W3) ***Rel. Local Commutativity***. The fields (anti-)commute at spacelike separations.

(W4) ***Vacuum State***. There exists a vector $|0\rangle$ in $\mathcal{H}$ satisfying the following conditions:

(i) $|0\rangle$ is Poincaré-invariant (***Invariance***).

(ii) $|0\rangle$ is cyclic for $\mathcal{H}$ (***Cyclicity***).

(iii) The spectrum of P^μ on the complement of $|0\rangle$ is confined to the forward lightcone (***Spectrum Condition***).

GQFT Axioms (Lévy-Leblond 1967)

(L1) ***Fields***. The fundamental dynamical variables are local field operators that act on a Hilbert space $\mathcal{H}$ of states.

(L2) ***Galilei-Symmetry***. $\mathcal{H}$ admits a unitary projective representation of the Galilei group.

(L3) ***Non-rel. Local Commutativity***. At equal times, the fields (anti-)commute for non-zero spatial separation.

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(i) $|0\rangle$ is Galilei-invariant (***Invariance***).

(ii) $|0\rangle$ is cyclic for mass sectors (***Cyclicity***).

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3. Axioms for QFTs: Free Particles

Poincaré Group
Projective irreducible representations labeled by mass and spin.

Galilei Group
Projective irreducible representations labeled by mass, internal energy, and spin.

Carriers of IRREPs represent states of "elementary systems". **(Wigner 1939)**

From IRREPs to free particles via Fock Space:

1. Identify carriers of IRREPs as elements $|q\rangle$ of *"single-particle" Hilbert space* $\mathcal{H}$.
2. Construct Fock space: $\mathcal{F} = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \mathcal{H}^3 \oplus \cdots$
3. Define *creation/annihilation operators* $a^\dagger(q)$, $a(q)$ on $\mathcal{F}$:

$$a^\dagger(q)|q_1 \ldots q_n\rangle = |q q_1 \ldots q_n\rangle,$$

$$a(q)|q_1 \ldots q_n\rangle = \sum_{r=1}^{n} (\pm 1)^{r+1} \delta(q - q_r)|q_1 \ldots q_{r-1} q_{r+1} \ldots q_n\rangle, \quad a(q)|0\rangle = 0.$$

4. Define *total number operator* N on $\mathcal{F}$: $N = \sum_q a^\dagger(q)\, a(q)$.

Eigenvectors of N are also eigenvectors of H. $\Rightarrow$ *They represent states with a definite number n of quanta with energies typical of n particles.*

3. Axioms for QFTs: Implications

1. The Reeh-Schlieder Theorem and Local Number Operators

"Local" Cyclicity of the Vacuum (Reeh & Schlieder 1961; Requardt 1982)

$\left(\begin{array}{c}\textit{spectrum condition}\\ \text{(W4iii/L4iii)}\end{array}\right) \Rightarrow \left(|0\rangle \text{ is } \textbf{\textit{cyclic}} \text{ for any local algebra of operators.}\right)$

Given any bounded spatiotemporal region $\mathcal{O}$ of (Minkowski or classical) spacetime, and the local algebra of operators $\mathfrak{A}(\mathcal{O})$ with support in $\mathcal{O}$, $\{\phi|0\rangle|\ \phi \in \mathfrak{A}(\mathcal{O})\}$ is dense in $\mathcal{H}$.

General Result (Bratelli & Robinson 1987)

Any cyclic vector for a von Neumann algebra is separating for its ***commutant.***

All operators that commute with elements of $\mathfrak{A}$.

Relativistic local commutativity (W3):

Commutant of $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')$, where $\mathcal{O}'$ is the ***causal complement*** of $\mathcal{O}$.

All points that are causally separated from points in $\mathcal{O}$.

3. Axioms for QFTs: Implications

1. The Reeh-Schlieder Theorem and Local Number Operators

"Local" Cyclicity of the Vacuum (Reeh & Schlieder 1961; Requardt 1982)

(*spectrum condition* (W4iii/L4iii)) $\Longrightarrow$ ($|0\rangle$ is ***cyclic*** for any local algebra of operators.)

Given any bounded spatiotemporal region $\mathcal{O}$ of (Minkowski or classical) spacetime, and the local algebra of operators $\mathfrak{A}(\mathcal{O})$ with support in $\mathcal{O}$, $\{\phi|0\rangle |\ \phi \in \mathfrak{A}(\mathcal{O})\}$ is dense in $\mathcal{H}$.

Separating Corollary (Streater & Wightman 1969)

- *local cyclicity*
- *rel. local comm.*
- *non-trivial $\mathcal{O}'$*

$\Longrightarrow$ ($|0\rangle$ is ***separating*** for any local algebra.)

Given any bounded region $\mathcal{O}$ of Minkowski spacetime and any operator $\phi \in \mathfrak{A}(\mathcal{O})$, if $\phi|0\rangle = 0$, then $\phi = 0$.

Upshot: No bounded region of Minkowski spacetime can contain annihilation operators!

Thus: ***Local number operators do not exist for RQFTs.***

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General Result (Bratelli & Robinson 1987)

Any cyclic vector for a von Neumann algebra is separating for its ***commutant.***

Non-relativistic local commutativity (L3):

Commutant of $\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')$, where $\mathcal{O}'$ is the *causal complement* of spatial $\mathcal{O}$.

But: For "spatial" local algebras, $|0\rangle$ is not cyclic!

And: For "spatiotemporal" local algebras, $\mathcal{O}'$ is trivial!

Hence: $|0\rangle$ is not separating for NQFTs.

3. Axioms for QFTs: Implications

1. The Reeh-Schlieder Theorem and Local Number Operators

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$$\left(\begin{array}{c}\textit{spectrum condition}\\ \text{(W4iii/L4iii)}\end{array}\right) \Longrightarrow \left(|0\rangle \text{ is } \underline{\textbf{\textit{cyclic}}} \text{ for any local algebra of operators.}\right)$$

Given any bounded spatiotemporal region $\mathcal{O}$ of (Minkowski or classical) spacetime, and the local algebra of operators $\mathfrak{A}(\mathcal{O})$ with support in $\mathcal{O}$, $\{\phi|0\rangle |\ \phi \in \mathfrak{A}(\mathcal{O})\}$ is dense in $\mathcal{H}$.

Cyclicity of the Vacuum for "spatial" local algebras:

- Hyperbolic differential operators are anti-local on spatial regions.
- Parabolic differential operators are not anti-local on spatial regions.

Moral: The existence of local number operators requires the existence of an absolute temporal metric.

3. Axioms for QFTs: Implications

<u>2a. The Unruh Effect and Unique Total Number Operators</u>

1. ***Existence*** of positive definite inner-product on $\mathcal{H}$ requires *global hyperbolicity* (Wald 1994). (Guarantees existence of a global time function by means of which to "split frequencies" of classical field equations.)

   <u>But</u>: Globally hyperbolic spacetimes may admit more than one global time function (*viz.*, timelike Killing vector field), and hence more than one way to "split the frequencies". (Unruh Effect in Minkowski spacetime.)

2. ***Uniqueness*** of global time function requires *absolute temporal metric.*

<u>Moral</u>: The existence of a unique total number operator for non-interacting QFTs requires the non-relativistic structure of classical spacetimes.

2b. Haag's Theorem and Total Number Operators

One cannot construct *unitarily equivalent* representations of the CCRs that describe both free and interacting relativistic quantum fields.

- ***Upshot***: In an interacting RQFT, the Fock space representation of free particles cannot be used to describe interacting particles.

- Moreover (Fraser forthcoming): Cannot construct an "interacting" Fock space representation by

  (a) Second-quantizing classical interacting fields.

  (b) Defining "interacting" creation/annihilation operators directly with respect to classical interacting fields.

- ***Conclusion***: *Total Number Operators (that can be interpreted as counting particle states) do not exist for interacting RQFTs.*

3. Axioms for QFTs: Implications

The Haag-Hall-Wightman (HHW) Theorem (Earman & Fraser 2006)

Let ϕ_1, ϕ_2 be local fields with unique vacuum states $|0_1\rangle$, $|0_2\rangle$.

(a) $\left(\begin{array}{l} \bullet \text{ IRREPs of the ETCCRs.} \\ \bullet \text{ Euclidean-invariance.} \\ \bullet \text{ Unitary equivalence at a given time.} \end{array}\right) \Longrightarrow \left(\text{Vac states const. multiples.}\right)$

(b) $\left(\begin{array}{l} \bullet \text{ Vac states const. multiples.} \\ \circ \text{ Poincaré-invariance.} \end{array}\right) \Longrightarrow \left(\text{If one field is free, then both are free.}\right)$

$$(\textit{Vacuum states are const. multiples}) \Leftrightarrow (\textit{No vacuum polarization})$$

Vacuum polarization

Suppose ϕ, ϕ_F are interacting and free fields with unique vacuum states $|0\rangle$, $|0_F\rangle$.

Let $H|0\rangle = 0$, $H_F|0_F\rangle = 0$, where $H = H_F + H_I$.

Then the interaction *polarizes the vacuum* just when $H|0_F\rangle \neq 0$.

3. Axioms for QFTs: Implications

The Haag-Hall-Wightman (HHW) Theorem (Earman & Fraser 2006)

Let ϕ_1, ϕ_2 be local fields with unique vacuum states $|0_1\rangle$, $|0_2\rangle$.

(a) $\left(\begin{array}{l} \bullet \text{ IRREPs of the ETCCRs.} \\ \bullet \text{ Euclidean-invariance.} \\ \bullet \text{ Unitary equivalence at a given time.} \end{array}\right) \Longrightarrow \left(\text{No vacuum polarization.}\right)$

(b) $\left(\begin{array}{l} \bullet \text{ No vacuum polarization.} \\ \circ \text{ Poincaré-invariance.} \end{array}\right) \Longrightarrow \left(\text{If one field is free, then both are free.}\right)$

Necessary conditions for the existence of interacting fields unitarily equivalent to free fields:

(i) Interaction polarizes the vacuum; or

(ii) Non-Poincaré-invariance.

Absolute temporal structure of classical spacetimes allows NQFTs to satisfy (ii) while denying (i).

3. Axioms for QFTs: Implications

Structure of extended Galilei Lie group: (Lévy-Leblond 1967)

- Generators: $(H, \mathbf{P}, \mathbf{K}, \mathbf{J}, M)$

  $[J_i, J_j] = i\varepsilon_{ijk}J_k,\quad [J_i, K_j] = i\varepsilon_{ijk}K_k,\quad [J_i, P_j] = i\varepsilon_{ijk}P_k,\quad [K_i, P_j] = iM\delta_{ij},$

  $[K_i, H] = iP_i,$

  $[J_i, H] = [K_i, K_j] = [P_i, P_i] = [P_i, H] = [H, M] = [J_i, M] = [P_i, M] = [K_i, M] = 0.$

- H nowhere occurs on RHS!

- Thus:
  - Let $(H_0, \mathbf{P}, \mathbf{K}, \mathbf{J}, M)$ be a "free" representation of extended Gal.
  - Let $H = H_0 + H_I$, where H_I is Gal-invariant.
  - Then free representation $(H_0, \mathbf{P}, \mathbf{K}, \mathbf{J}, M)$ is unitarily equivalent to "interacting" representation $(H, \mathbf{P}, \mathbf{K}, \mathbf{J}, M)$.
  - Thus if $H_0|0\rangle = 0$, then $H|0\rangle = 0$. (No vacuum polarization!)

3. Axioms for QFTs: Implications

Structure of Poincaré Lie group: (Lévy-Leblond 1967)

- Generators: $(H, \mathbf{P}, \mathbf{K}, \mathbf{J})$

  $[J_i, J_j] = i\varepsilon_{ijk}J_k$, $[J_i, K_j] = i\varepsilon_{ijk}K_k$, $[J_i, P_j] = i\varepsilon_{ijk}P_k$, $[K_i, P_j] = iH\delta_{ij}$, $[K_i, H] = iP_i$,
  $[K_i, K_j] = -i\varepsilon_{ijk}J_k$, $[J_i, H] = [P_i, H] = [H, H] = 0$

- H occurs on RHS of $[K_i, P_j] = iH\delta_{ij}$.

- Thus:
  - Let $(H_0, \mathbf{P}, \mathbf{K}, \mathbf{J})$ be a free representation of *Poincaré*.
  - Let $H = H_0 + H_I$, where H_I is *Poincaré*-invariant.
  - Then free representation $(H_0, \mathbf{P}, \mathbf{K}, \mathbf{J})$ is *not necessarily* unitarily equivalent to interacting representation $(H, \mathbf{P}, \mathbf{K}, \mathbf{J})$.
  - Thus, if $H_0|0\rangle = 0$, then *it need not be the case* that $H|0\rangle = 0$.

Moral: The existence of interacting NQFTs that do not polarize the vacuum reflects the absolute temporal structure of classical spacetimes.

4. Conclusion

Against the Received View:

- Local and (unique) total number operators reflect the absolute structure of classical spacetimes.
- Any concept of particle that requires the existence of local and (unique) total number operators is being informed by non-relativistic intuitions dependent on absolute concepts of time and space.

Options:

- Fault the Received View's pre-theoretic concept of particle (localizability and countability). Attempt to identify other conditions of adequacy for a particle concept that are compatible with the relativistic context.
- Fault the Received View's translation of its pre-theoretic concept of particle into the mathematical formalism. Attempt to identify representations of localizability and countability that are supported by the formalisms in which RQFTs, both free and interacting, are presented.
  - Investigate limiting relations between NQFTs and RQFTs as a way of determining the extent to which non-relativistic notions of particles (and fields) may be retained in RQFTs.