Non-Locality in Intrinsic Topologically Ordered Systems

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1. Two Types of Non-Locality.
2. Can Topological Non-Locality Entail Quantum Entanglement Non-Locality?
3. Intrinsic Topological Order and "Long-Range Entanglement".

Possibly, but current arguments are sketchy.

Even more sketchy...

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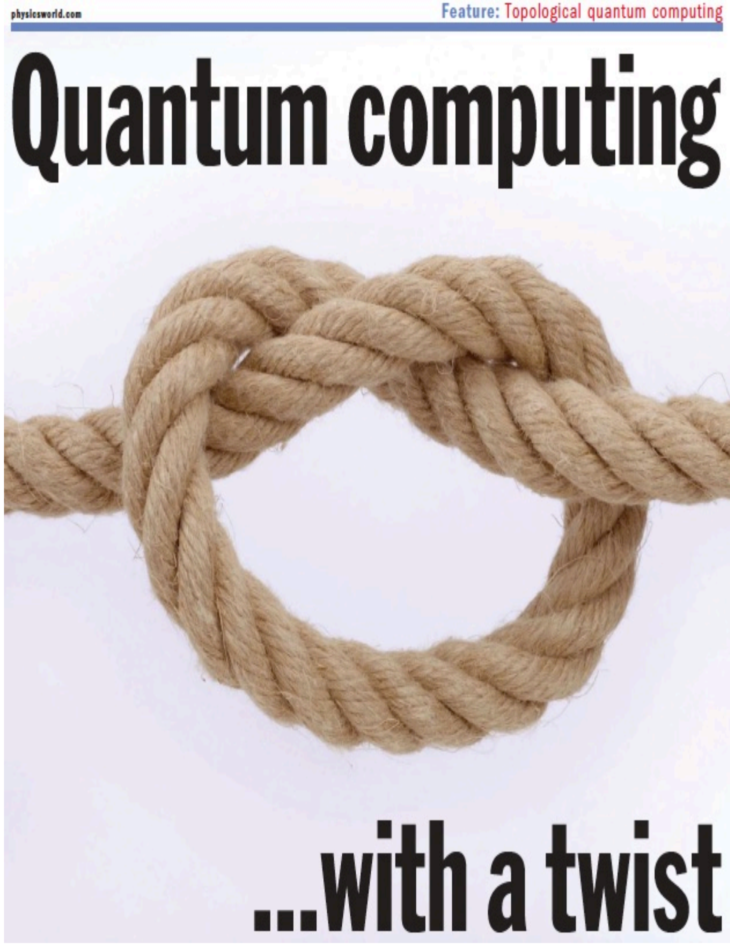
0. Motivation

Topological quantum computation

- *Goal*: To identify conditions that allow qubits to be encoded as *quantum entangled states* in a *topologically non-local* way that protects them against local errors.

- *Claim*: Intrinsic topologically ordered systems (like fractional quantum Hall systems), exhibit these conditions.

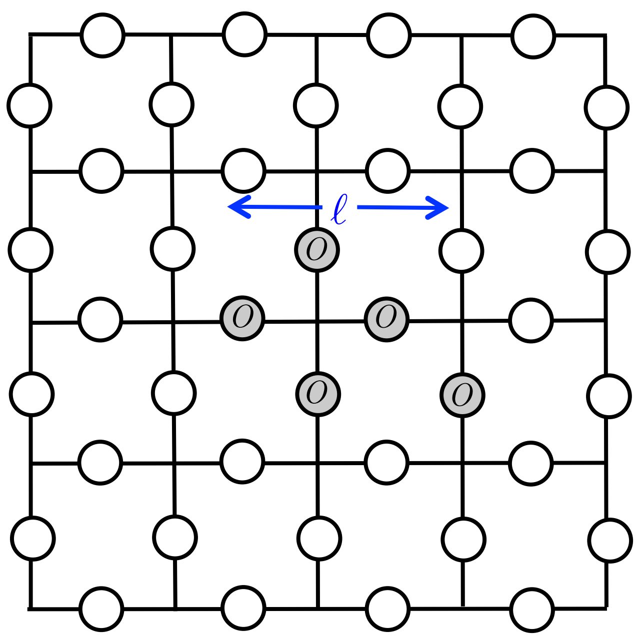


What is the relation between *quantum entanglement* and *topological non-locality* in such systems?

1. Two Types of Non-Locality

Localization = requirement that observables must be localized in finite regions of space.

Ex. Representing an observable by an *ℓ-local operator*: An operator O_{loc} that acts non-trivially only on a set X of lattice sites of diameter ℓ.



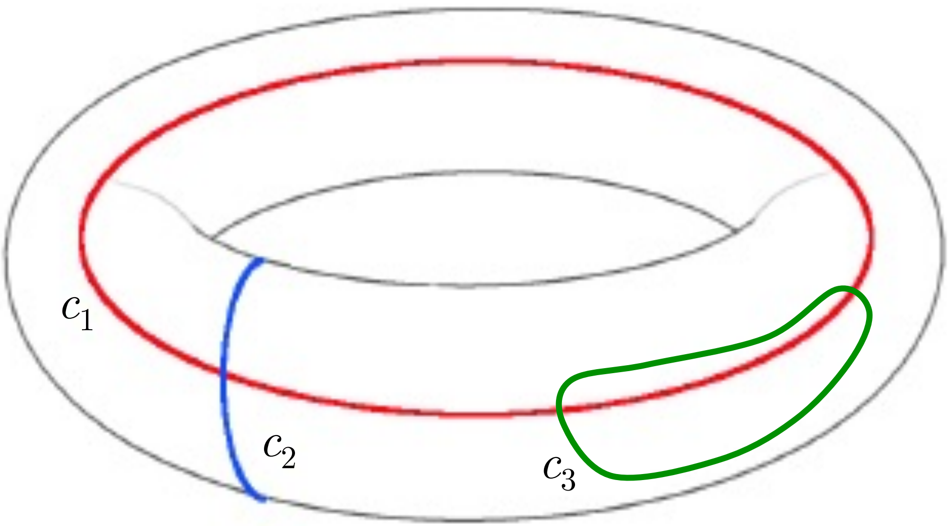
1 *subsystem per edge*,

$$\mathcal{H}^{(n)} = \bigotimes_{i=1}^{n} V_i$$

$$O_{loc} = \{\bigotimes_{i \notin X} I_i\} \otimes \{\bigotimes_{j \in X} O_j\}$$

<u>Topological non-locality</u>. Occurs when the observables of interest are, or encoded in, topological properties.

<u>*Ex.*</u> Representing an observable by a *non-contractible loop operator*: An operator O_{loop} that acts non-trivially only on lattice sites that form a non-contractible loop in a non-simply connected space.



- c_1, c_2 *are non-contractible.*
- c_3 *is contractible.*

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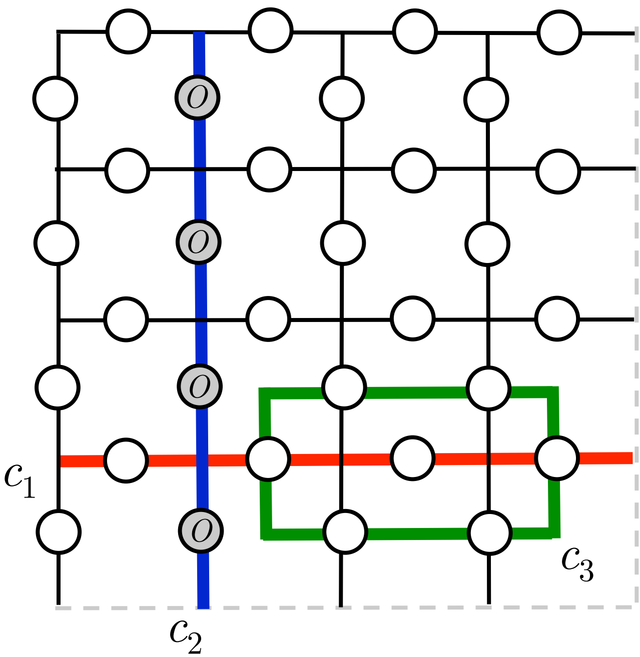


1 *subsystem per edge*,

$$\mathcal{H}^{(n)} = \bigotimes_{i=1}^{n} V_i$$

$$O_{loop} = \{\bigotimes_{i \notin c_2} I_i\} \otimes \{\bigotimes_{j \in c_2} O_j\}$$

Quantum entanglement non-locality. Occurs when the observables of interest exhibit *distant correlations* that *violate a Bell inequality*.

- Let A_X, B_Y be operators with support on lattice site sets X, Y.

(i) *Correlation* = statistical dependence

$\langle\psi|A_X B_Y|\psi\rangle \neq \langle\psi|A_X|\psi\rangle\langle\psi|B_Y|\psi\rangle$, for some ψ.

More than just this!

(ii) *Distant correlation* = no direct cause explanation

(a) $\langle\psi|A_X B_Y|\psi\rangle \neq \langle\psi|A_X|\psi\rangle\langle\psi|B_Y|\psi\rangle$, for some ψ.

(b) $\mathrm{dist}(X, Y) > v\Delta t$, v = bound on causal propagation

(iii) *Bell inequality-violating correlation* = no common cause explanation

(a) $\langle\psi|A_X B_Y|\psi\rangle \neq \langle\psi|A_X|\psi\rangle\langle\psi|B_Y|\psi\rangle$, for some ψ.

(b) $\langle\psi|A_X B_Y|\psi\rangle_\lambda \neq \langle\psi|A_X\rangle_\lambda\langle\psi|B_Y|\psi\rangle_\lambda$, for some ψ, and all λ.

conditional statistical dependence

Quantum entanglement non-locality. Occurs when the observables of interest exhibit *distant correlations* that *violate a Bell inequality*.

Ex. Some observables associated with maximally entangled 2-qubit Bell state for relevantly large separation distance between qubits.

$$\rho = (1/2)\left(|0_1 0_2\rangle + |1_1 1_2\rangle\right)\left(\langle 0_1 0_2| + \langle 1_1 1_2|\right)$$

Maximally entangled composite system state:

- $\rho_i = \mathrm{Tr}_j(\rho) = I/2, \qquad i, j = 1, 2$
- Relative to subsystem decomposition.

2. Can Top. Non-Locality $\Rightarrow$ Quant. Ent. Non-Locality?

(1) **Topological non-locality**. Occurs when the observables of interest are, or encoded in, topological properties.

(2) **Quantum entanglement non-locality**. Occurs when the observables of interest exhibit *distant correlations* that *violate a Bell inequality*.

(1) is a violation of localization.

(2) is consistent with localization.

Are there circumstances in which topological non-locality entails quantum entanglement non-locality?

A possible link?

Topological Quantum Order (ℓ-TQO) (Bravyi, *et al.* 2006)

An n-partite state $\psi_1 \in \mathcal{H}^{(n)}$ has ℓ-TQO *iff* there is another state ψ_2 orthogonal to it such that, for any ℓ-local operator O_{loc},

(i) $\langle\psi_1|O_{loc}|\psi_2\rangle = 0$, and — *can't map one into the other*

(ii) $\langle\psi_1|O_{loc}|\psi_1\rangle = \langle\psi_2|O_{loc}|\psi_2\rangle$. — *does the same thing to both*

- ℓ-TQO states are "locally indistinguishable". — *Topological non-locality, perhaps...*

- ℓ-TQO states are maximally entangled *with respect to* $\mathcal{H}^{(n)} = \mathcal{H}^{(\ell)} \otimes \mathcal{H}^{(n-\ell)}$. (Preskill 1999) — *Quantum entanglement non-locality!*

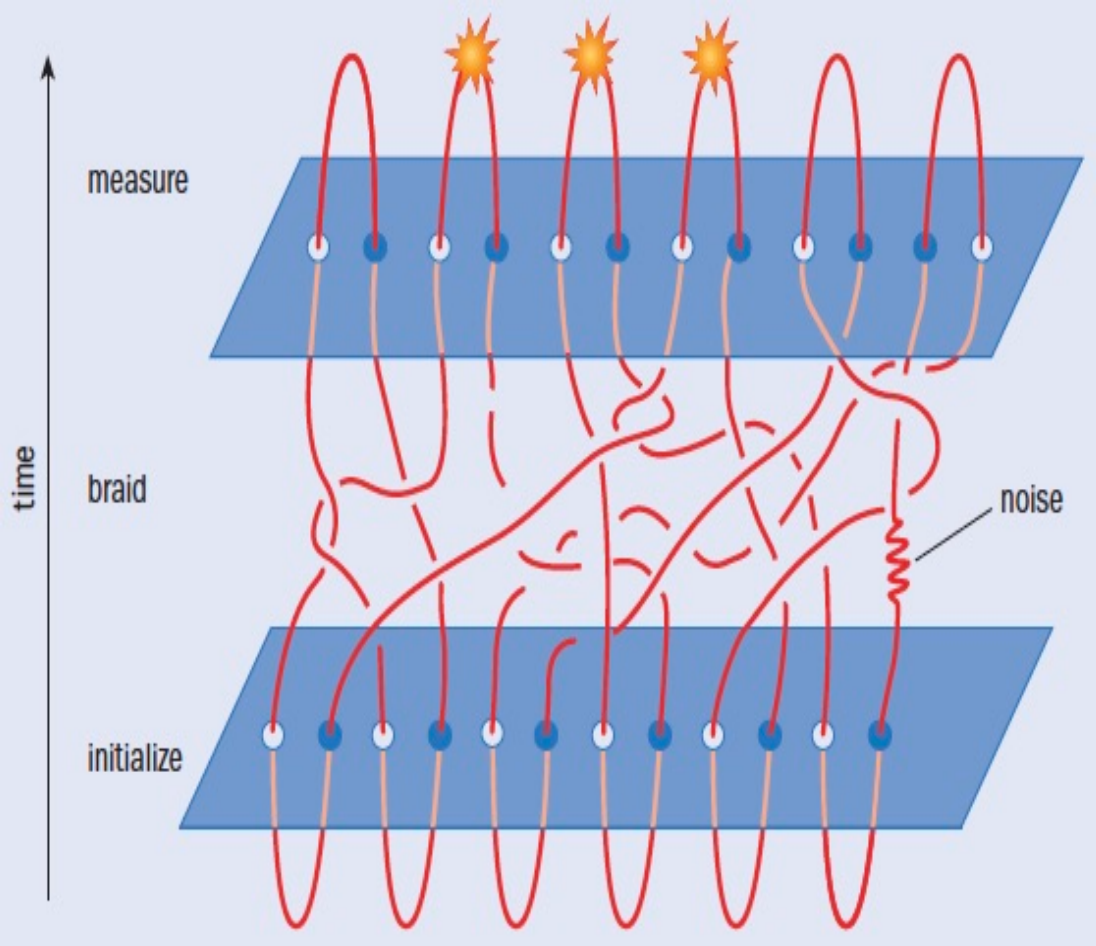
Suggestive but not definitive!

A stronger link: Intrinsic Topological Order (ITO)!

<u>*Goal*</u>: To perform operations on qubits that are immune to environmental "noise".

<u>*Proposal*</u>:

- Use 2-dim composite system that exhibits low-energy quasiparticle excitations that obey fractional stats.
- Represent operations on qubits by *braiding operations* on quasiparticles.
- In 2-dim, braids are topological invariants: immune to local perturbations!

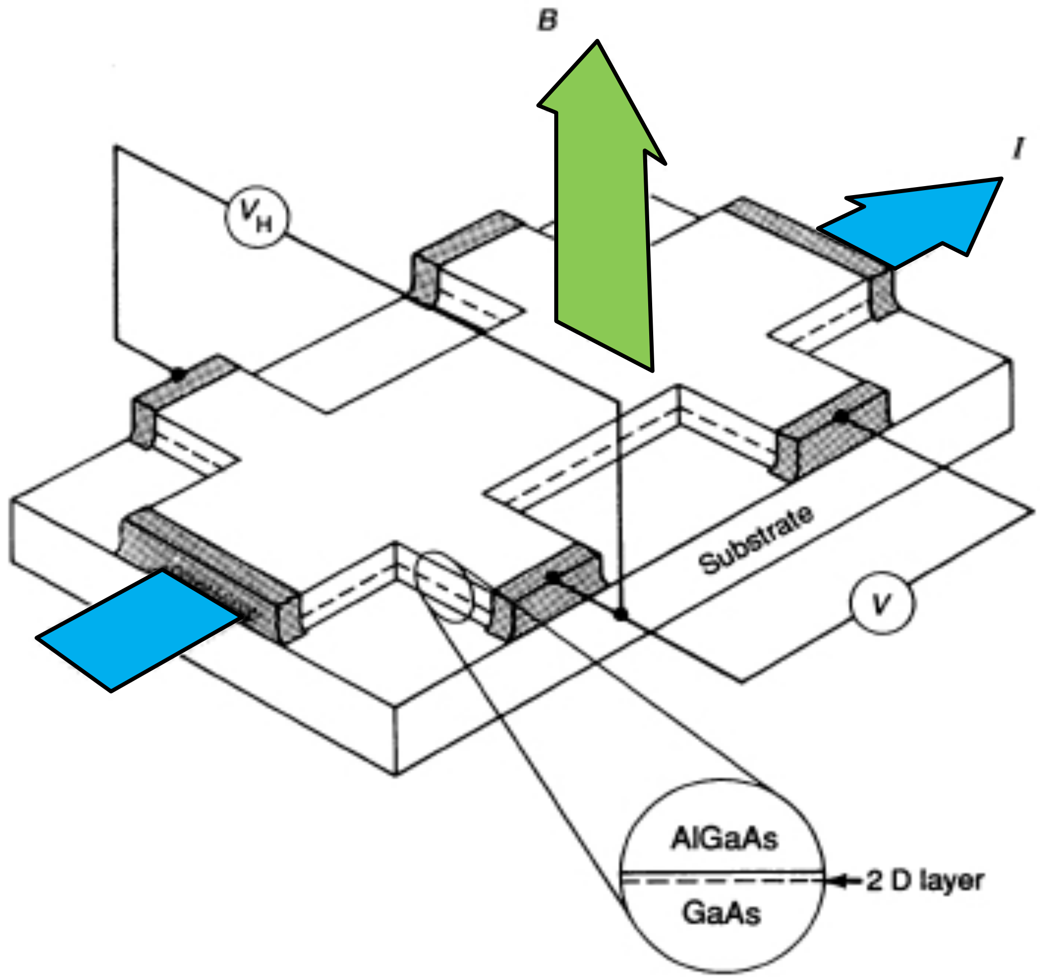


Simon (2010)

A stronger link: Intrinsic Topological Order (ITO)!

Paradigm Example: A fractional quantum Hall (FQH) system.

- 2-dim conductor in external magnetic field B.
- At low temps, longitudinal resistance vanishes, and transverse (Hall) resistance becomes quantized.
- *Prediction*: Low-energy anyonic excitations.
- *Other characteristics*:
  - Topology-dependent ground state degeneracy.
  - ℓ-TQO ground states.
  - Finite energy gap.



A stronger link?

Intrinsic Topological Order (ITO) (*e.g.*, Wen 2013)

A physical system possesses ITO just when:

(a) It exhibits a topology-dependent *ground state degeneracy*.

(b) It exhibits *anyonic low-energy excitations*.

(c) Its grounds states exhibit ℓ-TQO.

(d) Its ground states exhibit a *finite energy gap*.

- (a) & (b) are topological properties.

Topological non-locality, definitely!

- (a) & (b) entail (c)!
  (Kitaev 2003; Wen & Niu 1990)

Quantum entanglement non-locality!

But: (d) entails ground state correlations decay exponentially!

Exponential Clustering Theorem (Nachtergaele & Sims 2006)

Let ψ be a ground state with gap Δ of a reasonably local, non-relativistic Hamiltonian, and let A_X, B_Y be local observables with support on disjoint sets X, Y. Then

$$|\langle\psi|A_X B_Y|\psi\rangle - \langle\psi|A_X|\psi\rangle\langle\psi|B_Y|\psi\rangle| \leq C(\Delta)e^{-\mu\,\mathrm{dist}(X,Y)}$$

- *Implication*: ITO ground states approximate product states across sufficiently separated sets of lattice sites.
- *Apparent Conundrum*:
  Where did the quantum entanglement non-locality go?
- *Obvious Response*:
  It was never there to begin with!

- *Not So Obvious Response*:
  It's associated with "hidden long-range correlations"...

> "In FQH systems, the correlation of any local operators are short ranged. This seems to imply that FQH states are 'short sighted' and they cannot know the topology of space... However, the fact that ground-state degeneracy does depend on the topology of space implies that FQH states are not short sighted... So, despite the short-range correlations of any local operators, the FQH states must contain certain hidden long-range correlation[s]." (Chen, Gu, and Wen 2010)

- *In other words*: "Hidden long-range correlations" are required to explain the topology-dependent degeneracy of ITO ground states, *in the absence of an explanation underwritten by ground state correlations between local observables.*

3. ITO and Long-Range Entanglement

Claim: ITO ground states are "long-range entangled". (CGW 2010)

Short-range/long-range entanglement. A non-product state ψ is *short-range entangled iff* $\psi = U\psi_{\text{prod}}$, where U is a *local unitary evolution*, and ψ_{prod} is a product state. Otherwise, ψ is *long-range entangled.*

Lemma. Two gapped ground states ψ_1, ψ_2 are in the same *quantum phase iff* $\psi_1 = U\psi_2$, where U is a local unitary evolution.

Distinct ITO systems are in distinct quantum phases.

"Since a direct-product state is a state with trivial *topological order* [ITO], we see that a state with a short-range entanglement also has trivial *topological order*. This leads us to conclude that a nontrivial *topological order* is related to long-range entanglement." (CGW 2010)

CGW's Argument, Reconstructed

1. An n-partite product state w.r.t. an $\ell \times (n-\ell)$ decomp does not have ℓ-TQO, hence it is not an ITO ground state. *(Preskill 1999 & Def. of ITO.)*

"Since a direct-product state is a state with trivial *topological order* [ITO]..."

2. An SRE gapped ground state and a product gapped ground state belong to the same quantum phase. (*Def. of SRE & Lemma.*)

3. If two ground states are in the same quantum phase, then if one is not an ITO ground state, neither is the other. (*Assump.*)

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4. Thus, an SRE gapped ground state w.r.t. an $\ell \times (n-\ell)$ decomp cannot be an ITO ground state.

"...we see that a state with a short-range entanglement also has trivial *topological order*."

Granting Premise 3:

$$(\psi \text{ is an ITO ground state}) \Rightarrow (\psi \text{ is not SRE})$$

$$(\psi \text{ is LRE})$$

Concerns

(i) *What characterizes LRE correlations?*

- An ITO ground state cannot exhibit distant correlations between *local observables* that violate a Bell inequality.

  - *Exponential Clustering Theorem rules this out.*

- An ITO ground state cannot exhibit distant correlations between *topologically non-local observables*.

  - *Topologically non-local observables are metric-indepedent.*

What's left?

Concerns

(ii) *Ambiguity of entanglement.*

- An LRE state fails to be a product state in a *particular* way.
  - *Hence an LRE state exhibits correlations under particular conditions.*

- *Regardless*: Does it exhibit quantum entanglement non-locality?

- An FQH Hilbert space can be decomposed in empirically indistinguishable ways:
  - *electron subsystems*
  - *composite fermion subsystems*
  - *composite boson subsystems*

Which is the "physical" decomposition with respect to which we should define LRE?

Concerns

(iii) *Is LRE necessary to understand ITO systems?*

- Clustering theorem entails ITO ground state correlations between *local* observables decay exponentially.

  - *So: An explanation of the topology-dependent ground state degeneracy of an ITO system in terms of correlated local observables is problematic.*

  But: Why should we require such an explanation?

- *Diagnosis*: Assumption that an explanation of a topological property must take the form of a *localized* mechanistic (although perhaps non-causal) explanation.

  - *LRE as microphysical mechanism underlying ITO.* (Wen 2013)

  Why can't a mathematical/structural explanation suffice?

Explanandum: *Topology-dependent ground state degeneracy.*

Localized mechanistic explanation:

Appeal to a microphysical mechanism in the form of a collection of entities and activities:

- *Electrons in long-range entangled states.*

Structural explanation:

Appeal to the mathematical structure of a theory as a constraint on admissible phenomena:

- *Non-simply connected lattice space places constraints on kinematically possible states that an ITO system can be in.*

Conclusion

- Two distinct types of non-locality are claimed to be present in intrinsic topologically ordered systems:

  - Topological non-locality

  - Quantum entanglement non-locality

- Topological non-locality is certainly exhibited.

- Quantum entanglement non-locality is less clear.

  - Long-range entanglement doesn't necessarily entail it.

  - Long-range entanglement isn't necessary to account for the topological properties of ITO systems.

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