# Intertheoretic Implications of Non-Relativistic Quantum Field Theories


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1. NQFTs and Particles
2. Newtonian Quantum Gravity
3. Intertheoretic Relations

## 1. NQFTs and Particles

- **Relativistic quantum field theory (RQFT)** = A QFT invariant under the symmetries of a Lorentzian spacetime.

- **Non-relativistic quantum field theory (NQFT)** = A QFT invariant under the symmetries of a classical spacetime.

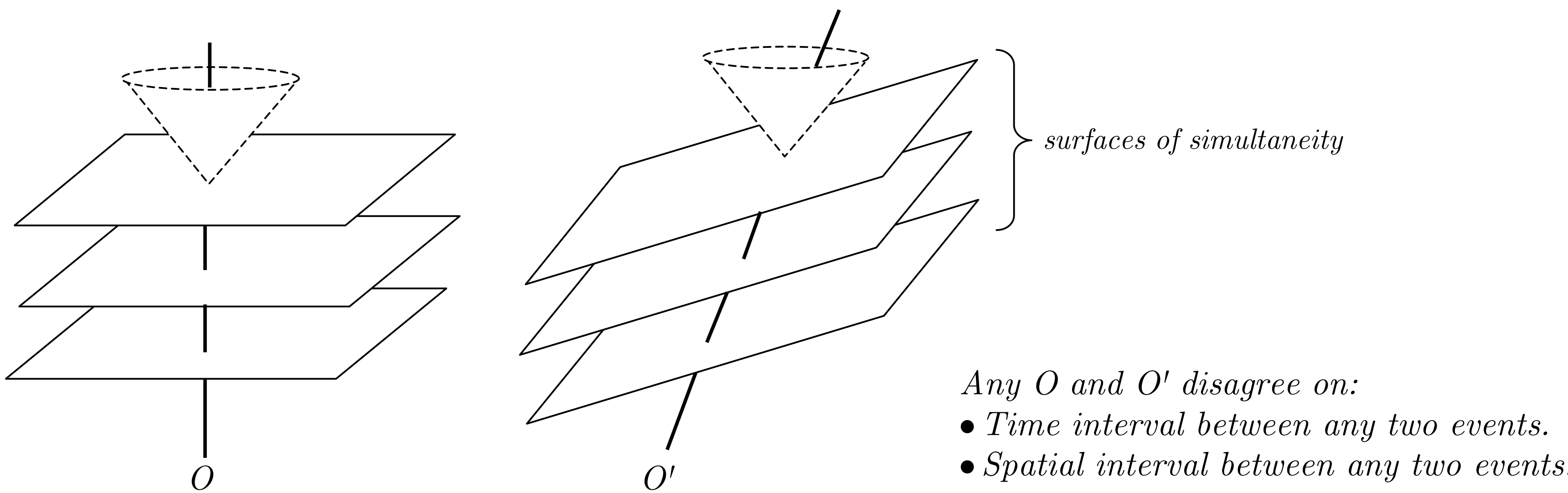
## 1. NQFTs and Particles

***Arena for RQFTs***: Lorentzian spacetime $(M, g_{ab})$.

- $g_{ab}$ - pseudo-Riemannian metric with Lorentzian signature (1, 3).
- $\nabla_a g_{bc} = 0$ for unique $\nabla_a$ (*compatibility*)

***Ex. 1***: Minkowski spacetime (*spatiotemporally flat*): $R^a{}_{bcd} = 0$.

- No unique way to separate time from space:




- Symmetry group generated by $\pounds_x g_{ab} = 0$. (*Poincaré group*)

## 1. NQFTs and Particles

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***Ex. 1***: Minkowski spacetime (*spatiotemporally flat*): $R^a{}_{bcd} = 0$.

***Ex. 2***: Vacuum Einstein spacetime (*Ricci flat*): $R_{ab} = 0$.

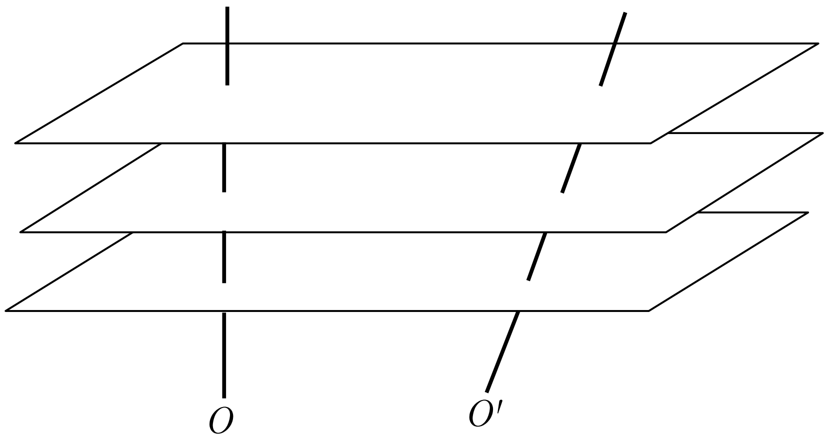
***Comparison:***

- *Different* metrical structure, *different* curvature, same metric signature (*i.e.*, "in the small", isomorphic to Minkowski spacetime).
- Different types of RQFTs, in flat (Minkowski) and curved Lorentzian spacetimes.

## 1. NQFTs and Particles

***Arena for NQFTs***: Classical spacetime $(M, h^{ab}, t_{ab}, \nabla_a)$.

- $h^{ab}$, $t_{ab}$ - degenerate metrics with signatures (0, 1, 1, 1) and (1, 0, 0, 0).
- $h^{ab}t_{ab} = 0 \quad (\textit{orthogonality})$
- $\nabla_c h^{ab} = 0 = \nabla_c t_{ab} \quad (\textit{compatibility}) \;\Rightarrow\;$ fails to uniquely determine $\nabla_a$
- Unique way exists to separate time from space:




*Any $O$ and $O'$ agree on:*
- *Time interval between any two events.*
- *Spatial interval between any two simultaneous events.*

- Symmetry group generated by $\pounds_x h^{ab} = \pounds_x t_{ab} = 0$.

## 1. NQFTs and Particles

***Arena for NQFTs***: Classical spacetime $(M, h^{ab}, t_{ab}, \nabla_a)$.

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***Ex. 1***: Neo-Newtonian spacetime (*spatiotemporally flat*): $R^a{}_{bcd} = 0$.

- ○ Symmetry group generated by $\pounds_x h_{ab} = \pounds_x t_{ab} = \pounds_x \Gamma^a{}_{bc} = 0$. (*Galilei group*)

***Ex. 2***: Maxwellian spacetime (*rotationally flat*): $R^{ab}{}_{cd} = 0$.

- ○ Symmetry group generated by $\pounds_x h_{ab} = \pounds_x t_{ab} = \pounds_x \Gamma^{ab}{}_c = 0$. (*Maxwell group*)

***Comparison:***

- *Same* metrical structure, *different* curvature.
- Different types of NQFTs, in flat (Neo-Newtonian) and curved classical spacetimes.

# 1. NQFTs and Particles

***Received View on Particles:***

(Arageorgis, Earman, Ruetsche 2003; Halvorson 2007; Halvorson and Clifton 2002; Fraser 2008)

*Necessary conditions for a particle interpretation:*

(A) The QFT must admit a Fock space formulation in which *local number operators* appear that can be interpreted as acting on a state of the system associated with a bounded region of spacetime and returning the number of particles in that region.

(B) The QFT must admit a *unique* Fock space formulation in which a *total number operator* appears that can be interpreted as acting on a state of the system and returning the total number of particles in that state.

## 1. NQFTs and Particles

***Claim 1***: Conditions (A) and (B) fail in RQFTs.

***Against (B) in RQFTs:***

- *Problem of Privilege:* RQFTs admit unitarily inequivalent Fock space representations of their CCRs.
- Minkowski spacetime exemption? Kay (1979): Minkowski quantization is unique up to unitary equivalence.
- *But*: The Unruh Effect (in one guise) says: "No!" (at least to some authors).
- *In any event:* Haag's Theorem says "No!" for realistic (interacting) RQFTs.

***Haag's Theorem*** $\Rightarrow$ *Representations of the CCRs for both a non-interacting and an interacting RQFT cannot be constructed so that they are unitarily equivalent at a given time.*

- Free particle total number operators cannot be used in interacting RQFTs.
- No consistent method for constructing "interacting" total number operators.

## 1. NQFTs and Particles

***Claim 1***: Conditions (A) and (B) fail in RQFTs.

***Against (A) in RQFTs:***

- *Separability Corollary (Streater & Wightman 2000)*: Let $\mathcal{A}$ be a local algebra of operators associated with a bounded region $\mathcal{O}$ of spacetime. If
  - (i) the vacuum state is cyclic for $\mathcal{A}$ ("local cyclicity");
  - (ii) $\mathcal{O}$ has non-trivial causal complement;
  - (iii) relativistic local commutativity holds;

  then the vacuum state is *separating* for $\mathcal{A}$.

  *For any $A \in \mathcal{A}$, if $A\Omega = 0$, then $A = 0$.*

- Reeh-Schlieder theorem secures (i) for Minkowski spacetime.
- Structure of Minkowski spacetime secures (ii).
- RQFTs satisfy (iii).
- *Thus*: Annihilation operators, hence number operators, cannot be defined in $\mathcal{A}$ for RQFTs in Minkowski spacetime.

# 1. NQFTs and Particles

*To what extent does the Separability Corollary hold for RQFTs in Lorentzian spacetimes in general?*

- Local cyclicity holds for RQFTs in ultrastatic and stationary Lorentzian spacetimes (Verch 1993, Bar 2000, Strohmeier 1999, 2000).

  > As soon as a classical field satisfies a certain hyperbolic partial differential equation, a state over the field algebra of the quantized theory, which is a ground- or KMS-state with respect to the group of time translations, has the Reeh-Schlieder property [*i.e.*, local cyclicity]. (Strohmeier 2000, pg. 106.)

- Is local cyclicity a generic feature of globally hyperbolic Lorentzian spacetimes?

- If so, then local cyclicity is not a generic feature of RQFTs in Lorentzian spacetimes:
  - Global hyperbolicity is not a necessary condition for the existence of an RQFT in a Lorentzian spacetime. (Fewster and Higuchi 1996.)

# 1. NQFTs and Particles

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- Is local cyclicity a generic feature of states analytic in the energy?

- Perhaps for RQFTs in Lorentzian spacetimes, but not for NQFTs in classical spacetimes:
  - Vacuum states for NQFTs are analytic but not locally cyclic for local algebras defined on spatial regions.

# 1. NQFTs and Particles

***Claim 2***: Conditions (A) and (B) hold in NQFTs due to the absolute temporal metric of classical spacetimes.

***Condition (A) in NQFTs:***

- *Non-relativistic local commutivity* ⇒ distinction between spatiotemporal local algebras and spatial local algebras.
- For spatiotemporal local algebra:
  - Requardt (1982) ⇒ Vacuum is locally cyclic.
  - *But*: Absolute temporal structure ⇒ Causal complement of $\mathcal{O}$ is trivial.
  - *Hence*: Vacuum is not separating.
- For spatial local algebras:
  - No local cyclicity result.
  - *Hence*: Vacuum is not separating.

## 1. NQFTs and Particles

*Why does local cyclicity fail for local algebras associated with spatial regions of a classical spacetime?*

- Let $\phi(t, \mathbf{x})$ be a positive-frequency solution to a well-posed PDE.
  - $\phi(t, \mathbf{x})$ is a boundary value of a holomorphic function.
- Let $\mathcal{S}$ be an open spatial region of spacetime.
  - If $\phi(t, \mathbf{x})$ vanishes on $\mathcal{S}$, then it vanishes in $D(\mathcal{S})$.

- *Case 1:* Hyperbolic PDE in Lorentzian spacetime.
  - $D(\mathcal{S})$ has non-zero temporal extent.
  - If $\phi$ vanishes on $\mathcal{S}$, then it vanishes in an open set in time, and thus everywhere (Edge of the Wedge theorem).
  - Thus: If $\phi \neq 0$, then it cannot vanish on $\mathcal{S}$. *Anti-locality for spatial regions.*

Segal and Goodman (1965)

- *Case 2:* Parabolic PDE in classical spacetime.
  - $D(\mathcal{S})$ has zero temporal extent.
  - If $\phi$ vanishes on $\mathcal{S}$, then it need not vanish in an open set in time.
  - Thus: If $\phi \neq 0$, then it can vanish on $\mathcal{S}$. *Anti-locality fails for spatial regions.*

## 1. NQFTs and Particles

***Claim 2***: Conditions (A) and (B) hold in NQFTs due to the absolute temporal metric of classical spacetimes.

***Condition (B) in NQFTs:***

- *No Problem of Privilege*: The absolute temporal metric guarantees a unique global time function on the spacetime, and this guarantees a unique means to construct a one-particle structure over the classical phase space (barring topological mutants).

# 1. NQFTs and Particles

***General Moral***:

To the extent that Conditions (A) and (B) require the existence of an absolute temporal metric, they are informed by a non-relativistic concept of time, and thus are inappropriate in informing interpretations of RQFTs.

## 2. Newtonian Quantum Gravity

I. Theories of Newtonian Gravity (NG) with a grav. potential field $\Phi$.

$(M,\ h^{ab},\ t_{ab},\ \nabla_a,\ \Phi,\ \rho)$

$h^{ab}t_{ab} = 0 = \nabla_c h^{ab} = \nabla_c t_{ab}$ *Orthogonality/compatibility*

$h^{ab}\nabla_{\mathrm{a}}\nabla_b\Phi = 4\pi G\rho$ *Poisson equation*

$\xi^a\nabla_a\xi^b = -h_{ab}\nabla_a\Phi$ *Equation of motion*

***Ex. 1:*** Neo-Newtonian NG

$R^a{}_{bcd} = 0$

***Ex. 2:*** "Island Universe" Neo-Newtonian NG

$R^a{}_{bcd} = 0, \quad \Phi \to 0 \text{ as } x^i \to \infty$

***Ex. 3:*** Maxwellian NG

$R^{ab}{}_{cd} = 0$

## 2. Newtonian Quantum Gravity

II. Theories of Newton-Cartan Gravity (NCG) that subsume $\Phi$ into connection. $(M,\ h^{ab},\ t_{ab},\ \nabla_a,\ \rho)$

$h^{ab}t_{ab} = 0 = \nabla_c h^{ab} = \nabla_a t_{ab}$ *Orthogonality/compatibility*

$R_{ab} = 4\pi G\rho t_{ab}$ *Generalized Poisson equation*

$\xi^a \nabla_a \xi^b = 0$ *Equation of motion*

***Ex. 1:*** Weak NCG ($1/c \to 0$ limit of GR)

$R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0$

***Ex. 2:*** Asymptotically spatially flat weak NCG (recovers Poisson equ.)

$R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0$, $R^{abcd} = 0$ at spatial infinity

***Ex. 3:*** Strong NCG (recovers Poisson equ.)

$R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0$, $R^{ab}{}_{cd} = 0$

## 2. Newtonian Quantum Gravity

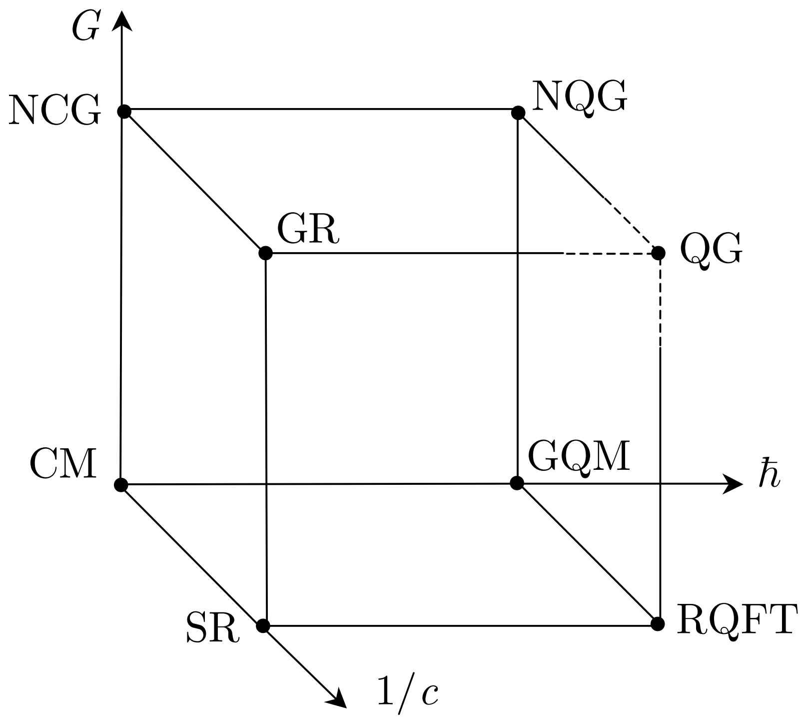
***Strong NCG***

- Christian (1997): constrained Hamiltonian system, reduced phase space.
- Unique one-parameter family of time evolution maps $\Rightarrow$ Unique Fock space quantization

***Newtonian Quantum Gravity (NQG)***

- Interacting (extended) Maxwell-invariant QFT of gravity in curved classical spacetime ("strong Newton-Cartan" spacetime).
- Satisfies Conditions (A) and (B).
- Gravitational degrees of freedom are *dynamic*: Compare with RQFTs in curved Lorentzian spacetimes.
- Gravitational degrees of freedom are *quantized*: Compare with semi-classical quantum gravity.

# 3. Intertheoretic Relations

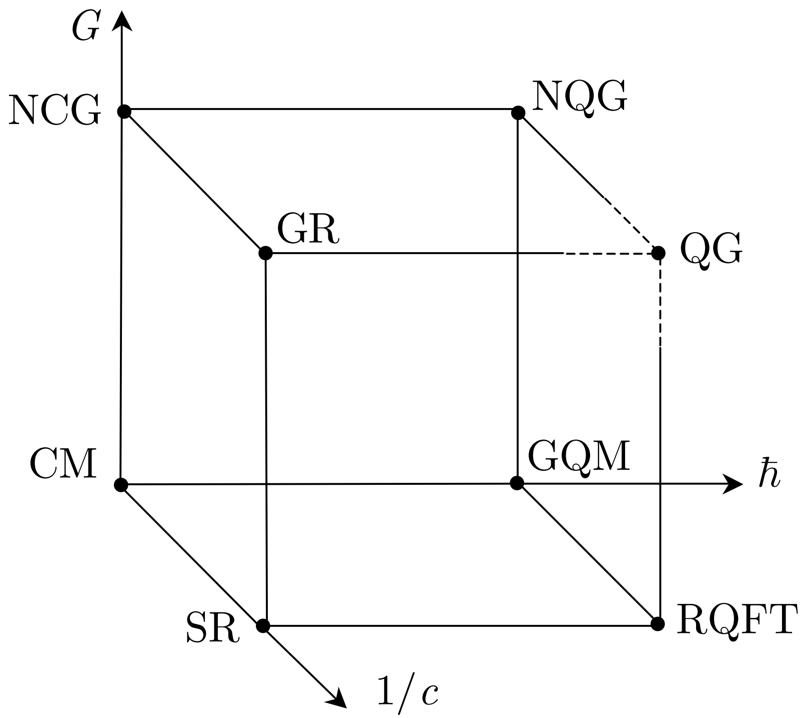



Christian (1997)

***1/c → 0 limit***

- Contraction of Poincaré Group? (Bacry & Levy-Leblond 1968)
- SR → CM, RQFT → GQM: Depends on dynamics. (Brown & Holland 2003)
- GR → NCG: No.

## 3. Intertheoretic Relations

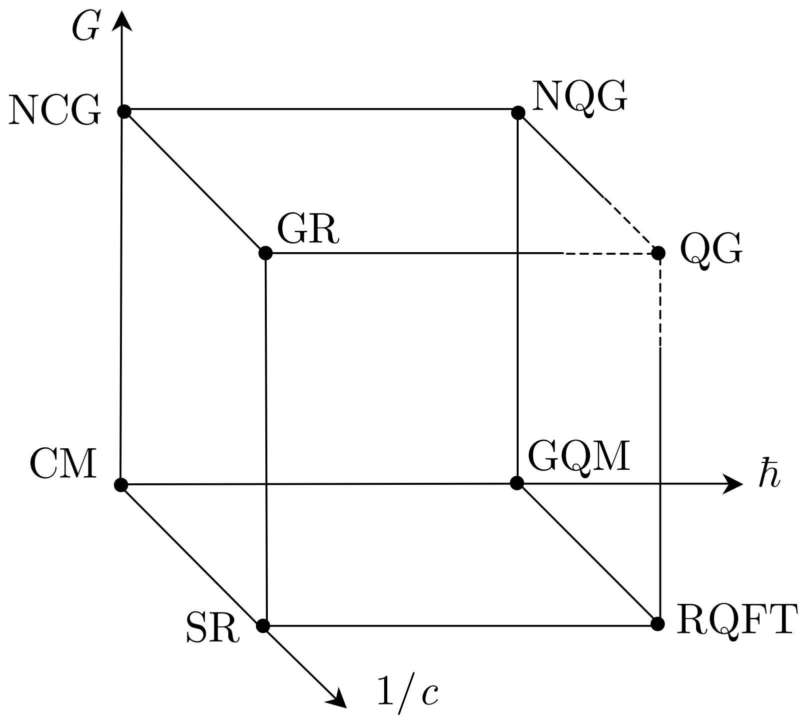



Christian (1997)

***G* → *0* limit: Ricci vs Riemann flatness**

- GR → SR: Vacuum Einstein spacetime *vs* Minkowski spacetime
- NCG → CM, NQG → GQM: Ricc-flat classical spacetime *vs* Neo-Newtonian spacetime

## 3. Intertheoretic Relations

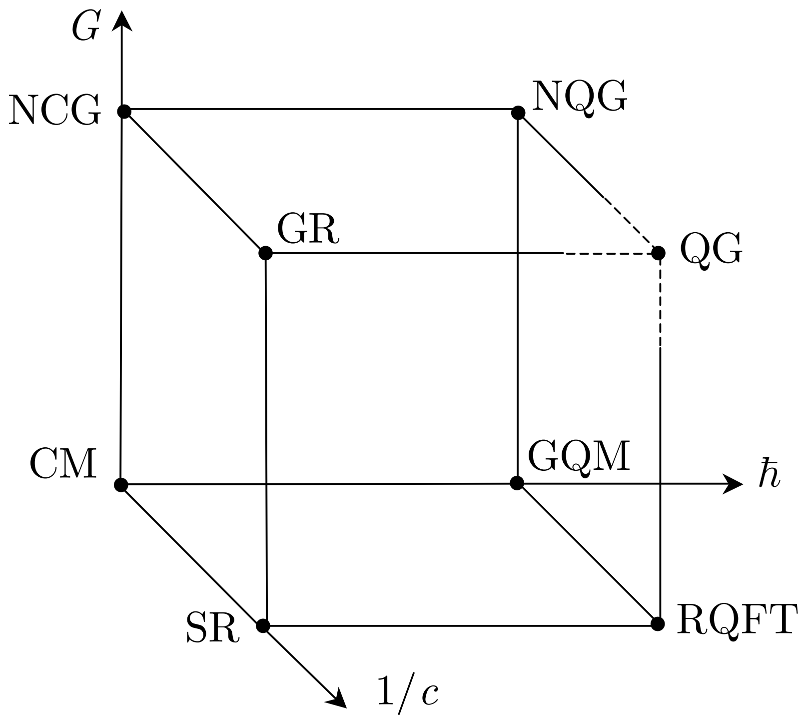



Christian (1997)

***$\hbar \to 0$ limit: Problem of Privilege***

- RQFT $\to$ SR: No unique (up to unitary equivalence) representation of CCRs.
- GQM $\to$ CM, NQG $\to$ NCG: No problem (barring topological mutants).

## 3. Intertheoretic Relations




Christian (1997)

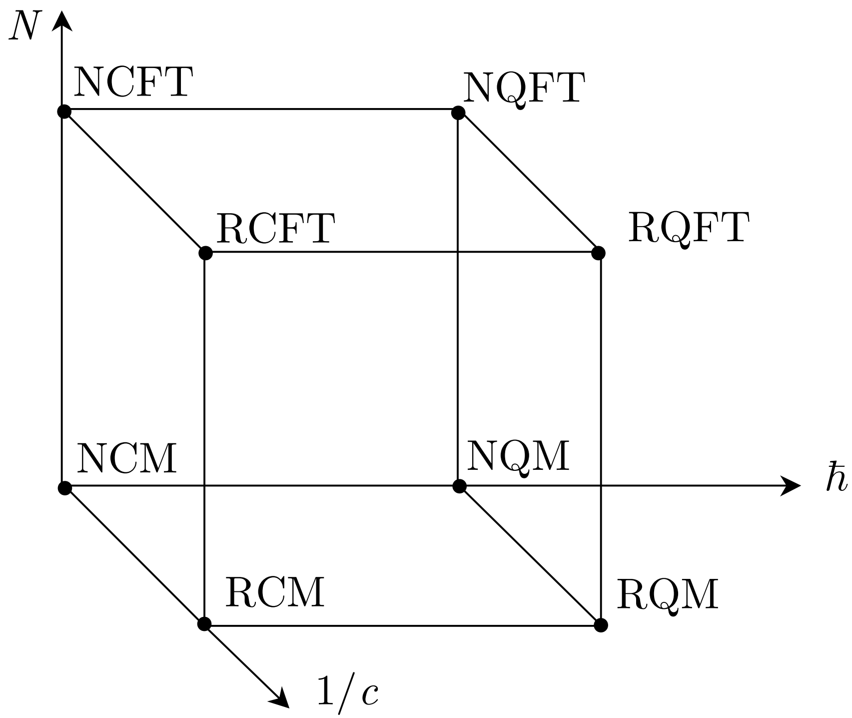
***<u>Structural Problem</u>***

- What is the referant of "GQM"? Where do NQFTs fit in?

***<u>Proposal: Add another axis for N = degrees of freedom</u>***

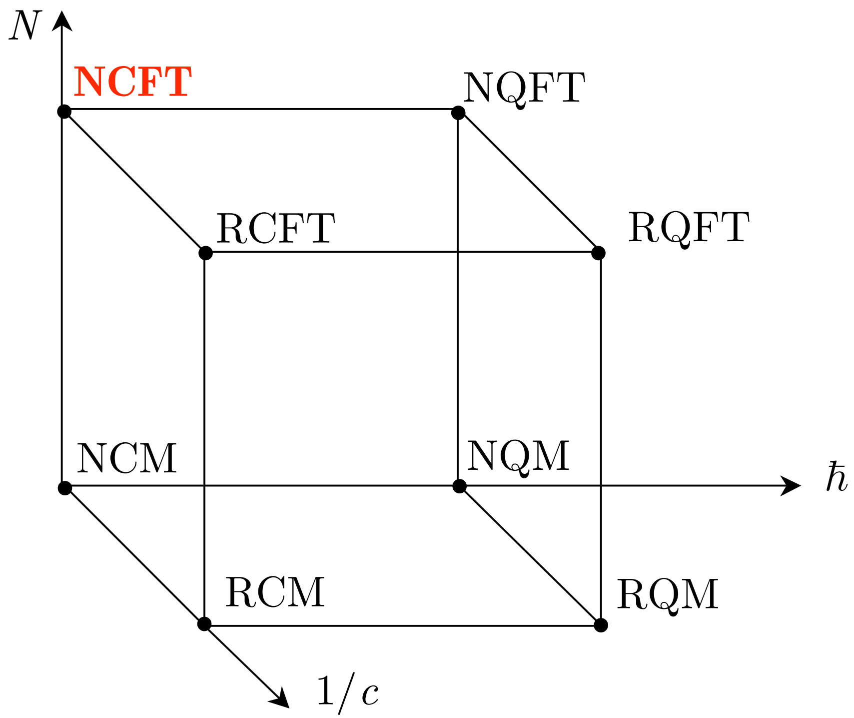
- Let "NQM" refer to non-relativistic finite-dimensional quantum theories of particle dynamics.
- Consider NQMs to be the $N \to 0$ limit of NQFTs.

# 3. Intertheoretic Relations




- Particle *vs* field theories ($N$ axis).
- Relativistic *vs* non-relativitic theories ($1/c$ axis).
- Gravitational *vs* non-gravitational theories ($G$ axis).
- Classical *vs* quantum theories ($\hbar$ axis).

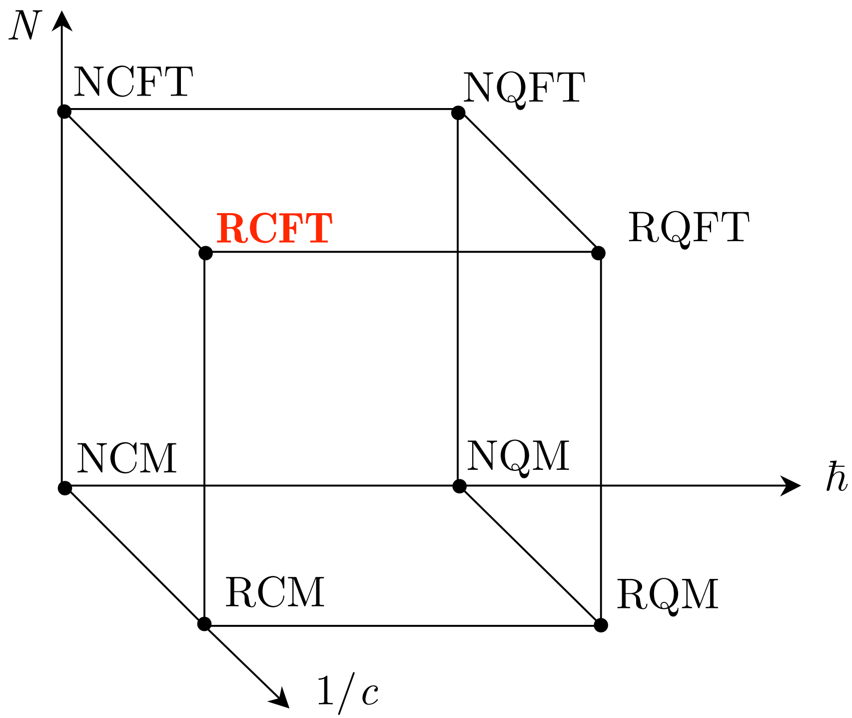
## 3. Intertheoretic Relations




***<u>Turning off G in field theories:</u>***

- Non-relativistic classical field theory of gravity $\rightarrow$ NCFT
- Asymptotically spatially flat NCG = "Island Universe" Neo-Newtonian NG
- $G \rightarrow 0$: Galilei-invariant classical field theory in Neo-Newtonian spacetime

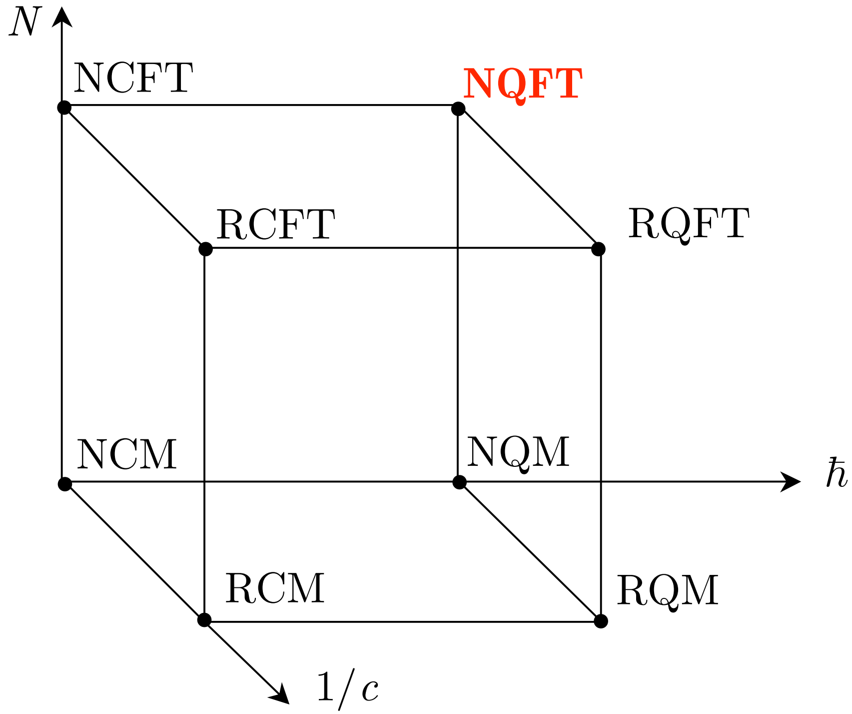
# 3. Intertheoretic Relations




***Turning off G in field theories:***

- Relativistic classical field theory of gravity $\longrightarrow$ RCFT
- GR
- $G \to 0$: Relativistic classical field theory in Ricci-flat Lorentzian spacetime

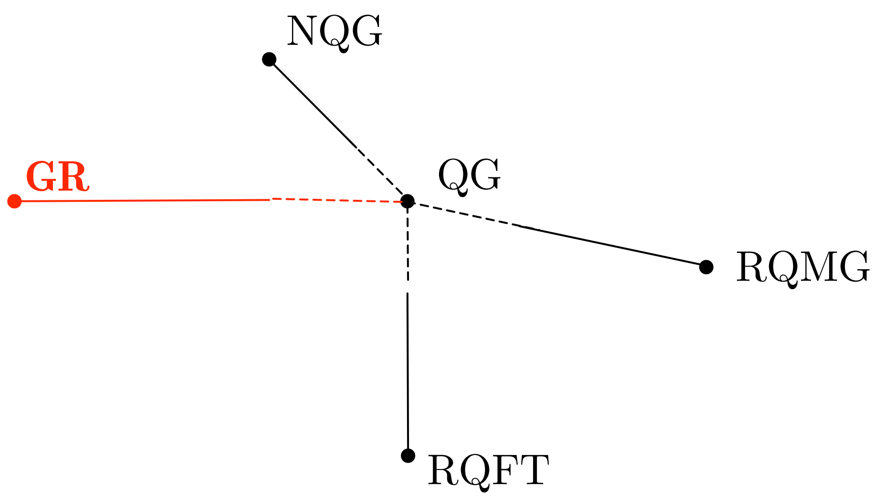
## 3. Intertheoretic Relations




***Turning off G in field theories:***

- Non-relativistic quantum field theory of gravity $\rightarrow$ NQFT
- NQG
- $G \rightarrow 0$: NQFT in Ricci-flat classical spacetime

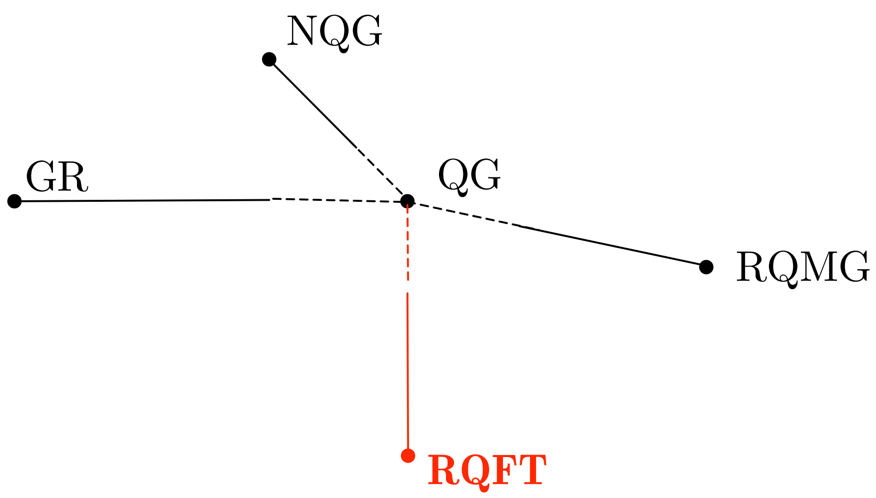
## 3. Intertheoretic Relations




***Turning on quantum gravity:***

- Quantizing GR.

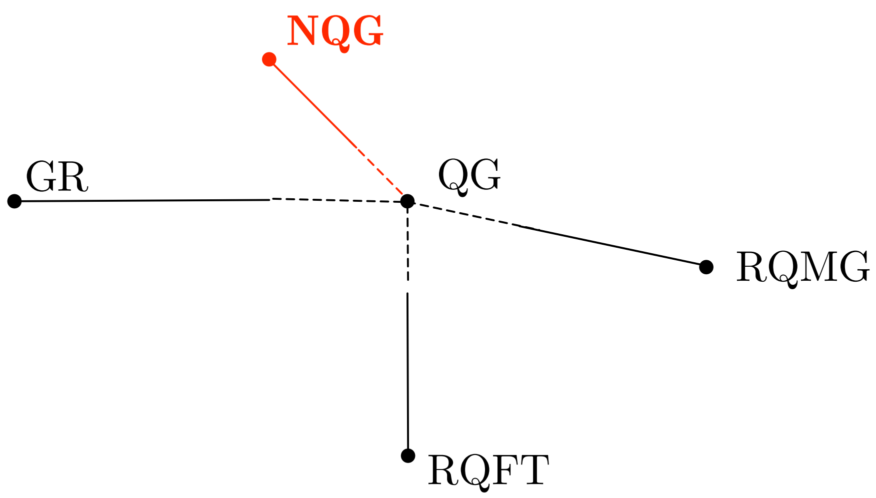
# 3. Intertheoretic Relations




***Turning on quantum gravity:***

- Quantizing GR.
- Turning on gravity in an RQFT.

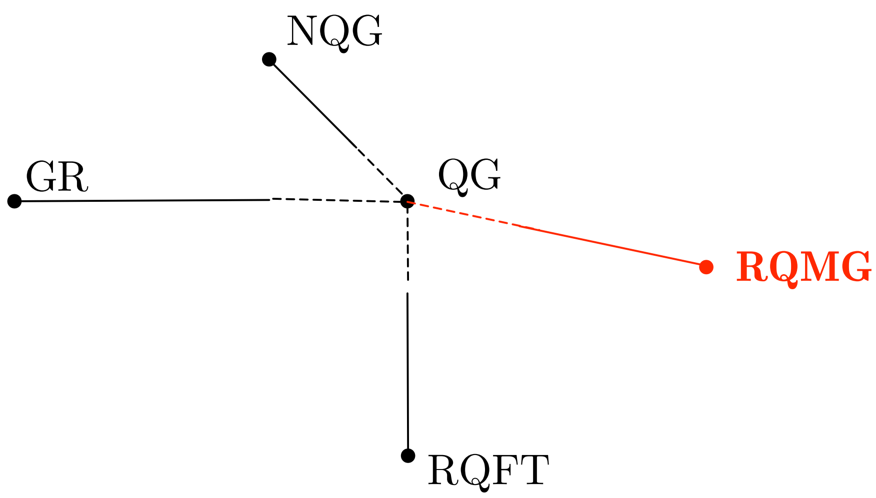
## 3. Intertheoretic Relations




***Turning on quantum gravity:***

- Quantizing GR.
- Turning on gravity in an RQFT.
- Relativizing NQG.

## 3. Intertheoretic Relations




***Turning on quantum gravity:***

- Quantizing GR.
- Turning on gravity in an RQFT.
- Relativizing NQG.
- Taking the "thermodynamic limit" of an RQMG.