Motivating Structural Realist Interpretations of Spacetime

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1. Realism With Respect to What?
2. Dynamical *vs.* Kinematical Structure
3. Is Structure Jones-Underdetermined?
4. What is Structure?

1. Realism With Respect to *What*?

Scientific Realism

Successful theories should be *interpreted literally*: we should take them at their face-value.

"Jones" Underdetermination (Jones 1991)

- Successful theories typically admit alternative mathematical formulations that disagree at the level of ontology.

- ***Thus***: What should scientific realists be realists about?

1. Realism With Respect to *What*?

General Relativity

Tensor models:

$$(M,\ g_{ab})$$

differentiable manifold

metric field satisfying Einstein equations

$\xleftrightarrow{\text{1-1}}$

points of M correspond to maximal ideals of $\mathcal{R}^\infty$

Einstein algebra (EA) models

$$(\mathcal{R}^\infty,\ \mathcal{R},\ g)$$

commutative ring

subring of $\mathcal{R}^\infty$ isomorphic to $\mathbb{R}$

multilinear map on space of derivations of $(\mathcal{R}^\infty, \mathcal{R})$ and its dual, satisfying Einstein equations

- *Idea*: Reconstruct M as collection of maximal ideals of commutative ring $C^\infty(M)$ of smooth functions on M.
- *Different Indivs.-based Ontologies*: points *vs.* ideals
- *Common Structure:* Differentiable structure

1. Realism With Respect to *What*?

Claim: Manifold points *kinematically matter*; maximal ideals do not.

Consider: GR with asymptotic boundary conditions.

- *Asymptotically flat GR.*
- *GR with singularities.*

Tensor Models

- Replace *manifold* M with *manifold with boundary* $M' = M \cup \partial M$.
- (M, g_{ab}) is $\mathrm{Diff}(M)$-invariant.
- (M', g_{ab}) is $\mathrm{Diff}_c(M)$-invariant, but *not necessarily* $\mathrm{Diff}(M)$-invariant.

diffeomorphisms on M with compact support $\approx$ *"local" diffeomorphisms*

- No morphisms that preserve *both* M and M'.
- M and M' belong to *different* categories.

1. Realism With Respect to *What*?

Claim: Manifold points *kinematically matter*; maximal ideals do not.

Consider: GR with asymptotic boundary conditions.

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- *GR with singularies.*

Einstein Algebra (EA) Models

(1) Replace *ring* $\mathcal{R}^\infty \cong C^\infty(M)$ with *sheaf* $\mathcal{R}^\infty_{Asymp} \cong C^\infty(M')$.

(2) Replace *Einstein algebra* $(\mathcal{R}^\infty, g)$ with *sheaf of Einstein algebras* $(\mathcal{R}^\infty_{Asymp}, g)$.

- $(\mathcal{R}^\infty, g)$ and $(\mathcal{R}^\infty_{Asymp}, g)$ are objects in a *single category*: the category of "structured spaces" (Heller & Sasin 1995).
- There are morphisms that preserve the structure of *both* $(\mathcal{R}^\infty, g)$ and $(\mathcal{R}^\infty_{Asymp}, g)$.

1. Realism With Respect to *What*?

Claim: Manifold points *kinematically matter*; maximal ideals do not.

Consider: GR with asymptotic boundary conditions.

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- *GR with singularies.*

Upshot:

- Kinematical structure of EA models: *"global" differentiable structure* (morphisms preserving $(\mathcal{R}^\infty, g)$, $(\mathcal{R}^\infty_{Asymp}, g)$).
- Kinematical structure of tensor models: *"local" differentiable structure* (differentiable structure at p depends on whether $p \in M$ or $p \in \partial M$).

1. Realism With Respect to *What*?

General Relativity

Tensor models:

Twistor models

$$(M,\ g_{ab}{}^{ASD}) \xleftrightarrow{\text{1-1}} (\mathcal{P},\ \tau,\ \rho)$$

anti-self-dual metric satisfying vacuum Einstein equations

Non-linear Graviton Penrose Transformation (Penrose 1976)

"curved" twistor space

differential forms on $\mathcal{P}$

- *Idea*: Modify correspondence between Minkowski spacetime and twistor space "infinitesimally" for curved spacetimes.
- *Different Indivs.-based Ontologies*: Points *vs.* twistors.
- *Common Structure:* Conformal structure

General Relativity

Tensor models: **Geometric algebra (GA) models**

$$(M,\ g_{ab},\ (e_\mu)^a) \quad \xleftrightarrow{\text{1-1}} \quad (\mathcal{D},\ \bar{h},\ \Omega)$$

global tetrad field

Gauge Theory of Gravity (Lansby et al 1998)

Dirac algebra

displacement gauge field (function) on $\mathcal{D}$

rotation gauge field (function) on $\mathcal{D}$

- *Idea*: Impose displacement and rotation gauge invariance on a matter Lagrangian defined on $\mathcal{D}$.
- *Different Indivs.-based Ontologies*: Points *vs.* multivectors.
- *Common Structure:* Metrical structure

2. Dynamical *vs.* Kinematical Structure

Dynamically Equivalent Models of GR:

- Tensor models *sans* b.c.'s $\cong$ EA models
- Tensor models $w/$b.c.'s $\cong$ EA models
- ASD tensor models $\cong$ Twistor models
- Tensor models $w/$global tetrad fields $\cong$ GA models

Kinematically Distinct Models of GR:

- Tensor models: local differentiable structure
- EA models: global differentiable structure
- Twistor models: conformal structure
- GA models: metrical structure

2. Dynamical *vs.* Kinematical Structure

Sector	Models	Spacetime Structure	Dynamical Structure
GR *sans* b.c.'s	tensor	local differentiable	$(M,\ g_{ab}) \cong (\mathcal{R}^{\infty},\ g)$
	EA	global differentiable	
GR w/b.c.'s	tensor	local differentiable	$(M \cup \partial M,\ g_{ab}) \cong (\mathcal{R}^{\infty}_{Asymp},\ g)$
	EA	global differentiable	
ASD-GR	tensor	local differentiable	$(M,\ {g_{ab}}^{ASD}) \cong (\mathcal{P},\ \tau,\ \rho)$
	twistor	conformal	
tetrad-GR	tensor	local differentiable	$(M,\ g_{ab},\ (e_{\mu})^{a}) \cong (\mathcal{D},\ \bar{h},\ \Omega)$
	GA	metrical	

2. Dynamical *vs.* Kinematical Structure

Suggests a Distinction Between:

(A) A structural realist interpretation of a theory.

An ontological commitment to the dynamical structure associated with the theory.

(B) A structural realist interpretation of spacetime as described by a particular formulation of a given theory.

An interpretation of spacetime as given by the kinematical structure associated with that formulation of the theory.

3. Is Structure Jones-Underdetermined?

Claim: Jones Underdetermination cannot motivate structural realism.

Why?

Alternative formalisms disagree

(i) At the level of individuals

AND

(ii) At the level of structure

THUS

Not only are individuals-based interpretations of a single theory underdetermined; so are structural realist interpretations.

3. Is Structure Jones-Underdetermined?

Pooley (2006, pp. 87-88):

"Consider a model of a theory of Newtonian gravitation formulated using an action-at-a-distance force and an ***empirically equivalent*** model of the Newton-Cartan formulation of the theory. There is no (primitive) element of the second model which is structurally isomorphic to the flat inertial connection of the first model, and there are no (primitive) elements of the first model which are structurally isomorphic to the gravitational potential field, or the non-flat inertial structure of the second. Clearly a more sophisticated notion of structure is needed if it is to be something common to models of both formulations of the theory."

3. Is Structure Jones-Underdetermined?

But:

- Not *really* an example of Jones Underdetermination: Two ways of formulating the *same* theory in the *same* (tensor) formalism.
- *Can* a single theory admit distinct formulations in a *single* formalism that differ at the level of structure?

3. Is Structure Jones-Underdetermined?

I. Theories of Newtonian Gravity (NG) with a grav. potential field Φ.
$(M,\ h^{ab},\ t_{ab},\ \nabla_a,\ \Phi,\ \rho)$

$h^{ab}t_{ab} = 0 = \nabla_c h^{ab} = \nabla_c t_{ab}$	*Orthogonality/compatibility*
$h^{ab}\nabla_a\nabla_b\Phi = 4\pi G\rho$	*Poisson equation*
$\xi^a\nabla_a\xi^b = -h^{ab}\nabla_a\Phi$	*Equation of motion*

	Spacetime	**Dynamical**
Ex. 1:		
Neo-Newtonian NG $R^a{}_{bcd} = 0$	$\mathfrak{gal}$	$\widetilde{\mathfrak{max}}$
Ex. 2:		
Island Universe Neo-Newtonian NG $R^a{}_{bcd} = 0, \quad \Phi \to 0$ as $x^i \to \infty$	$\mathfrak{gal}$	$\mathfrak{gal}$ $\Phi \mapsto \Phi + \varphi(t)$
Ex. 3:		
Maxwellian NG $R^{ab}{}_{cd} = 0$	$\mathfrak{max}$	$\widetilde{\mathfrak{max}}$

3. Is Structure Jones-Underdetermined?

II. Theories of Newton-Cartan Gravity (NCG) that subsume ϕ into connection. $(M,\ h^{ab},\ t_{ab},\ \nabla_a,\ \rho)$

$h^{ab}t_{ab} = 0 = \nabla_c h^{ab} = \nabla_c t_{ab}$	*Orthogonality/compatibility*
$R_{ab} = 4\pi G\rho t_{ab}$	*Generalized Poisson equation*
$\xi^a \nabla_a \xi^b = 0$	*Equation of motion*

	Spacetime	**Dynamical**
Ex. 1: Weak NCG $R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0$	$\widetilde{\mathfrak{leib}}$	$\widetilde{\mathfrak{leib}}$
Ex. 2: Asymptotically spatially flat weak NCG $R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0,\quad R^{abcd} = 0$ at spatial infinity	$\mathfrak{gal}$	$\mathfrak{gal}$ $\Phi \mapsto \Phi + \varphi(t)$
Ex. 3: Strong NCG $R^{[a}{}_{[b}{}^{c]}{}_{d]} = 0,\quad R^{ab}{}_{cd} = 0$	$\widetilde{\mathfrak{max}}$	$\widetilde{\mathfrak{max}}$

3. Is Structure Jones-Underdetermined?

Theory	**ST symmetries**	**Dynamical symmetries**
Neo-Newtonian NG	$\mathfrak{gal}$	$\widetilde{\mathfrak{max}}$
Island Universe Neo-Newt NG	$\mathfrak{gal}$	$\mathfrak{gal}$ and $\Phi \mapsto \Phi + \varphi(t)$
Maxwellian NG	$\mathfrak{max}$	$\widetilde{\mathfrak{max}}$
Weak NCG	$\widetilde{\mathfrak{leib}}$	$\widetilde{\mathfrak{leib}}$
Asymp. spatially flat Weak NCG	$\mathfrak{gal}$	$\mathfrak{gal}$ and $\Phi \mapsto \Phi + \varphi(t)$
Strong NCG	$\widetilde{\mathfrak{max}}$	$\widetilde{\mathfrak{max}}$

Empirically Indistinguishable Theories

(a) Island Universe Neo-Newt NG; Asymp. spatially flat Weak NCG

(b) Neo-Newt NG; Max NG; Strong NCG

Example of Underdetermination of Structure?

Case (a)? No:

- Possess same spacetime symmetries, hence make the same ontological commitments *vis-a-vis* spacetime structure.

3. Is Structure Jones-Underdetermined?

Theory	**ST symmetries**	**Dynamical symmetries**
Neo-Newtonian NG	$\mathfrak{gal}$	$\widetilde{\mathfrak{max}}$
Island Universe Neo-Newt NG	$\mathfrak{gal}$	$\mathfrak{gal}$ and $\Phi \mapsto \Phi + \varphi(t)$
Maxwellian NG	$\mathfrak{max}$	$\widetilde{\mathfrak{max}}$
Weak NCG	$\widetilde{\mathfrak{leib}}$	$\widetilde{\mathfrak{leib}}$
Asymp. spatially flat Weak NCG	$\mathfrak{gal}$	$\mathfrak{gal}$ and $\Phi \mapsto \Phi + \varphi(t)$
Strong NCG	$\widetilde{\mathfrak{max}}$	$\widetilde{\mathfrak{max}}$

Empirically Indistinguishable Theories

(a) Island Universe Neo-Newt NG; Asymp. spatially flat Weak NCG

(b) Neo-Newt NG; Max NG; Strong NCG

Example of Underdetermination of Structure?

Case (b)?

- All *disagree* on their kinematical structure; *i.e.*, what they take to be the structure of spacetime.
- *But*: All *agree* on their dynamical structure.

3. Is Structure Jones-Underdetermined?

Claim 1. Structural realist interpretations of different formulations of a single theory do not suffer from underdetermination of *dynamical structure*.

Claim 2. Structural realist interpretations of spacetime as represented by a particular formulation of a theory *are* underdetermined.

But: Underdetermination of spacetime structure:

- Has no affect on *current* empirical adequacy of the theory.
- Is susceptible to *future* empirical tests:

Extensions of GR to Quantum Gravity:

Twistors $\Rightarrow$ Twistor approach to QG.

Einstein algebras $\Rightarrow$ Heller & Sasin (1999) QG.

Geometric algebra $\Rightarrow$ Background dependent QG.

4. What is Structure?

Radical Ontic Structural Realism (French & Ladyman 2003)

Structure consists of relations devoid of *relata*.

Untenable?

Set-theoretically, perhaps so.

- Suppose *structure* = *isomorphism class of structured sets* = $[\{X, R_i\}]$.
- A (*binary*) *relation* R on X is a subset of $X \times X$, the set of all ordered pairs (x_1, x_2), $x_1, x_2 \in X$.
- An *ordered pair* (x_1, x_2) is the set $\{x_1, \{x_1, x_2\}\}$.
- Ineliminable reference to elements ("*relata*") of a set.

4. What is Structure?

Radical Ontic Structural Realism (French & Ladyman 2003)

Structure consists of relations devoid of *relata*.

Untenable?

Category-theoretically, perhaps not.

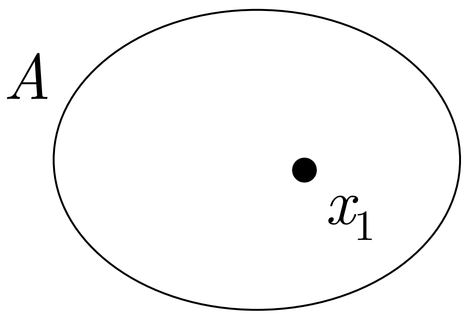
- Suppose *structure* = *object in a category*.
- "Internal" constituents of an object ("elements") referred to purely in terms of "external" objects and morphisms.

4. What is Structure?

- An object $\mathbf{1}$ of a category $\boldsymbol{\mathcal{C}}$ is a *terminal object* of $\boldsymbol{\mathcal{C}}$ if for each object X of $\boldsymbol{\mathcal{C}}$, there is exactly one $\boldsymbol{\mathcal{C}}$-morphism $X \to \mathbf{1}$.
- An *element* of an object A in a category $\boldsymbol{\mathcal{C}}$ is a morphism $\mathbf{1} \to A$, where $\mathbf{1}$ is the terminal object in $\boldsymbol{\mathcal{C}}$.

Set Theory

Primitives: sets, $\in$



$$x_1 \in A$$

Category Theory

Primitives: objects, morphisms

$$\mathbf{1} \xrightarrow{x_1} A$$

4. What is Structure?

- The *Cartesian product* of an object X with itself is an object P, together with a pair of morphisms $p_1 : P \to X$, $p_2 : P \to X$ such that, for any arbitrary object T with morphisms $f_1 : T \to X$, $f_2 : T \to X$, there is exactly one morphism $f : T \to P$ for which $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.

$$
\begin{tikzcd}
 & & X \\
T \arrow[r, "f"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & P \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & & X
\end{tikzcd}
$$

- External probe T, f_1, f_2, f encodes internal pair structure of P.

4. What is Structure?

Objection: Elimination of relata in name only.

- Where set theory sees "elements", category theory sees "morphisms from the terminal object".
- "*No relations without relata*" becomes "*No objects without morphisms*".

Response

- Manifold points have correlates in EA, but ultimately these correlates are surplus in EA models of GR.
- Similarly, set-theoretic *relata* have correlates in category theory, but ultimately these correlates are surplus.
  - Category-theoretic objects need not be structured sets.
  - Such objects have roles to play in articulating relevant notions of structure in physics. Baez (2006)

4. What is Structure?

What the Category-theoretic Radical Ontic Structural Realist must do:

- Provide rationale for fundamentality of category theory over set theory. (Pedroso 2008)
- Provide category-theoretic formulations of scientific theories that do not presuppose **Set**. (Döring & Isham 2008; Isham and Butterfield 2000; Baez 2006)
- Identify the relevant notion of structure in category-theoretic terms.
  - Distinguish between ***kinematical*** structure and ***dynamical*** structure in category-theoretic terms.

4. What is Structure?

How to do physics in category theory: (Baez 2006)

Given a theory T,

- "Kinematics" of T = objects A, B, ... in category $\boldsymbol{\mathcal{C}}$.
- Dynamics of T = morphisms $f\colon A \to B$, $g : C \to D$, ... in $\boldsymbol{\mathcal{C}}$.

Ex1: Classical physics

$\boldsymbol{\mathcal{C}} = \mathbf{Symp}$

objects = symplectic manifolds (classical phase spaces)

morphisms = symplectic transformations

Ex2: Quantum physics

$\boldsymbol{\mathcal{C}} = \mathbf{Hilb}$

objects = Hilbert spaces (quantum phase spaces)

morphisms = bounded linear operators

4. What is Structure?

How to do physics in category theory: (Baez 2006)

Given a theory T,

- "Kinematics" of $T =$ objects A, B, ... in category $\boldsymbol{\mathcal{C}}$.
- Dynamics of $T =$ morphisms $f : A \to B$, $g : C \to D$, ... in $\boldsymbol{\mathcal{C}}$.

However:

- The "kinematics" here describes space of *dynamically* possible states.
- Distinction between *kinematically* possible states and *dynamically* possible states.

4. What is Structure?

How to do field-theoretic physics: (Belot 2007)

A field theory consists of $(\mathcal{K}, \Delta)$, where

(i) $\mathcal{K}$ is the space of *kinematically possible* fields $\phi : M \to W$, where M is a differentiable manifold (*viz.*, spacetime) and W is an appropriate space in which the fields take values.

(ii) Δ is a set of differential equations consisting of *independent* variables (parametrizing M) and *dependent* variables (parametrizing W).

- Define space of *dynamically possible* fields $\mathcal{S} = \{\phi_0 \in \mathcal{K} : \phi_0$ is a solution of $\Delta\}$.
- ***Dynamical structure*** = Structure of $\mathcal{S}$.
- ***Kinematical structure*** = Structure of independent variables in Δ.

4. What is Structure?

Kinematically Distinct Models of GR:

(a) Tensor models: local differentiable structure

(b) EA models: global differentiable structure

(c) Twistor models: conformal structure

(d) GA models: metrical structure

Category-theoretic translations:

(a) (i) **Man** = category of smooth manifolds

(ii) **Manb** = category of smooth manifolds with boundary

(b) **Struc** = category of structured spaces (Heller and Sasin 1995)

(c) **Twist** = category of (curved) twistor spaces

(d) $\mathbf{Cliff}_{(\mathbf{1,3})}$ = category of Dirac algebras

4. What is Structure?

Sector	**Models**	**Spacetime Structure**		**Dynamical Structure**	
GR *sans* b.c.'s	tensor	local differentiable	**Man**	$(M,\ g_{ab}) \cong (\mathcal{R}^\infty,\ g)$	$\mathbf{Symp}_1$
	EA	global differentiable	**Struc**		
GR w/b.c.'s	tensor	local differentiable	**Manb**	$(M \cup \partial M,\ g_{ab}) \cong (\mathcal{R}^\infty_{Asymp},\ g)$	$\mathbf{Symp}_2$
	EA	global differentiable	**Struc**		
ASD-GR	tensor	local differentiable	**Man**	$(M,\ {g_{ab}}^{ASD}) \cong (\mathcal{P},\ \tau,\ \rho)$	$\mathbf{Symp}_3$
	twistor	conformal	**Twist**		
tetrad-GR	tensor	local differentiable	**Man**	$(M,\ g_{ab},\ (e_\mu)^a) \cong (\mathcal{D},\ \overline{h},\ \Omega)$	$\mathbf{Symp}_4$
	GA	metrical	$\mathbf{Cliff}_{(1,3)}$		

- $\mathbf{Symp} \supset \mathbf{Symp}_i \cong \mathcal{S}$ for given $(\mathcal{K},\ \Delta)$.

5. Conclusion

- Dynamical *vs.* kinematical structure.
- Motivates distinction between structural realist interpretations of a theory *vs.* structural realist interpretations of spacetime as described by a theory.
- Blunts Jones Underdetermination arguments against structural realism.
- Can be articulated in category-theoretic terms.