What Explains the Spin–Statistics Connection?

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0. What is the Spin-Statistics Connection?

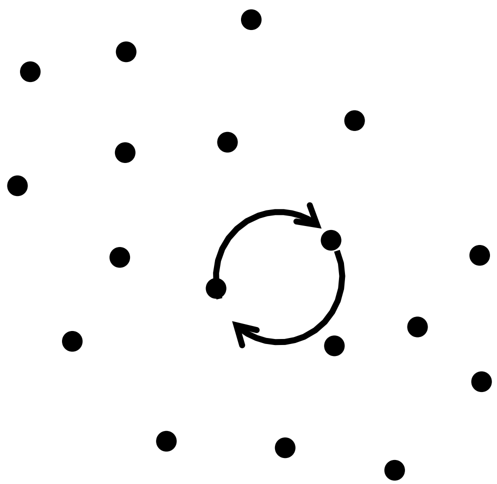
Spin-statistics connection (SSC):

i. Physical systems that obey BE statistics possess integer spin.

ii. Physical systems that obey FD statistics possess half-integer spin.

Statistics in terms of a multiparticle system:

- Describes how the system behaves under single-particle exchanges.
- Two important features that can characterize such exchanges:
  (a) *Permutation Invariance*
  (b) *Exclusion Principle*
- BE statistics is characterized by (a).
  FD statistics is characterized by both (a) and (b).



0. What is the Spin-Statistics Connection?

Permutation invariance = multiparticle system "remains unchanged" under exchange of any two particles.

One way to encode this:

- Let $|\Phi\rangle$ represent a multiparticle state.
- Let $|\Phi'\rangle$ be obtained from $|\Phi\rangle$ by exchanging any two of its single-particle substate vectors.

$|\Phi\rangle$ is *permutation invariant* just when $\langle\Phi|A|\Phi\rangle = \langle\Phi'|A|\Phi'\rangle$ for any operator A representing an observable quantity.

Two ways to guarantee this:

$|\Phi'\rangle = |\Phi\rangle$ (*symmetric state vector*)

$|\Phi'\rangle = -|\Phi\rangle$ (*antisymmetric state vector*)

0. What is the Spin-Statistics Connection?

Exclusion Principle = no two particles can be in the same single-particle substate.

Motivation: Pauli's (1925) exclusion principle for electrons

Electrons in an atom are characterized by:

- Energy n — $n = 1, 2, \ldots$
- Orbital angular momentum ℓ — $\ell = 0, 1, 2, \ldots (n-1)$
- z-component of orbital ang. mo. m_ℓ — $m_\ell = -\ell, \ldots 0, \ldots, \ell$
- Spin m_s — $m_s = -1/2, +1/2$

- *So*: The state of an electron is characterized by four values (n, ℓ, m_ℓ, m_s).

Pauli's Exclusion Principle (1925):
No two electrons can be in the same state; *i.e.*, no two electrons can have all the same values of (n, ℓ, m_ℓ, m_s).

0. What is the Spin-Statistics Connection?

		n:	1	2		3			4			
Z	*Element*	ℓ:	0	0	1	0	1	2	0	1	2	3
1	H hydrogen		1									
2	He helium		2									
3	Li lithium		2	1								
4	Be beryllium		2	2								
5	B boron		2	2	1							
6	C carbon		2	2	2							
7	N nitrogen		2	2	3							
8	O oxygen		2	2	4							
9	F fluorine		2	2	5							
10	Ne neon		2	2	6							

Energy shells

K shell $(n = 1)$

L shell $(n = 2)$

M shell $(n = 3)$

N shell $(n = 4)$

etc.

Orbitals

s orbital $(\ell = 0)$

p orbital $(\ell = 1)$

d orbital $(\ell = 2)$

f orbital $(\ell = 3)$

etc.

Example: The 3 electrons in a lithium atom are characterized by:

$(1, 0, 0, +1/2), (1, 0, 0, -1/2), (2, 0, 0, +1/2).$

0. What is the Spin-Statistics Connection?

One way to encode the exclusion principle:

Consider some examples of 2-particle multiparticle state vectors

1. $|\Phi_1\rangle = \sqrt{1/2}\{|\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle\}$ *symmetric*
2. $|\Phi_2\rangle = \sqrt{1/2}\{|\phi\rangle|\psi\rangle - |\psi\rangle|\phi\rangle\}$ *anti-symmetric*
3. $|\Phi_3\rangle = \sqrt{1/2}|\phi\rangle|\phi\rangle + \sqrt{1/4}\{|\phi\rangle|\psi\rangle - |\psi\rangle|\phi\rangle\}$ *non-symmetric*

- Suppose we allow particles 1 and 2 to be in identical states.
  - *Let* $|\psi\rangle = |\phi\rangle$ *in* 1-3.
- *Then*: The anti-symmetric multiparticle state vector vanishes! The others don't.
- *Suggests*: Use anti-symmetric vectors to represent the states of a multiparticle system that is both Permutation Invariant and obeys the Exclusion Principle.

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Encoding Statistics: Normal Commutation Relations (NCR) for particles

"Bosonic" multiparticle state

- Permutation Invariance.
- *Require*: $[a(\mathbf{p}),\, a^{\dagger}(\mathbf{p}')] = \delta(\mathbf{p}-\mathbf{p}')$ (symmetric state)

"Fermionic" multiparticle state

- Permutation Invariant and Exclusion Principle.
- *Require*: $\{a(\mathbf{p}),\, a^{\dagger}(\mathbf{p}')\} = \delta(\mathbf{p}-\mathbf{p}')$ (antisymmetric state)

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Encoding Statistics: Normal Commutation Relations (NCR) for fields

Relativistic case

Either $[\phi^\dagger(x),\, \phi(x')] = 0$, or $\{\phi^\dagger(x),\, \phi(x')\} = 0$, for spacelike $(x - x')$.

Non-relativistic case

Either $[\phi^\dagger(\mathbf{x}, t),\, \phi(\mathbf{x}', t)] = 0$, or $\{\phi^\dagger(\mathbf{x}, t),\, \phi(\mathbf{x}', t)\} = 0$, for $(\mathbf{x} - \mathbf{x}') \neq 0$.

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Spin:

- Describes how a physical system behaves under rotations.
- For rotation through angle 2π:
  - State of integer-spin system picks up phase $+1$.
  - State of half-integer-spin system picks up phase -1.

- Werner *et al.* (1975): observation of half-integer-spin behavior in neutron interferometry experiment.

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i. Physical systems that obey BE statistics possess integer spin.

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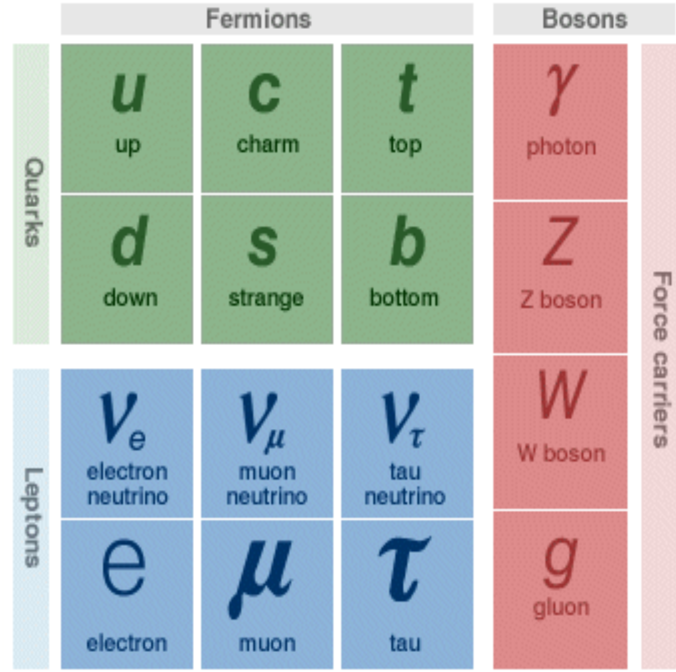
Encoding Spin: group representations

- Relativistic integer/half-integer spin state = carrier of true/double-valued representation of Poincaré group $\mathcal{P}$.

- Non-relativistic integer/half-integer spin state = carrier of true/double-valued representation of Galilei group $\mathcal{G}$.

$$\mathcal{P} = \underset{\substack{\big\uparrow \text{ 2:1} \\ SL(2,\mathbb{C})}}{SO(1,3)} \times \mathbb{R}^{1,3} \qquad \mathcal{G} = (\underset{\substack{\big\uparrow \text{ 2:1} \\ SU(2)}}{SO(3)} \times \mathbb{R}^3) \times (\mathbb{R}^1 \times \mathbb{R}^3)$$

1. Why SSC Needs an Explanation



Explanatory Power of SSC:

- RQFTs represent fundamental matter systems as possessing half-integer spin, and interacting via the exchange of integer spin carriers of fundamental gauge fields ("forces").

- <u>*Thus*</u>: On the basis of our best theories of matter, SSC entails that fundamental matter systems must be fermions that obey the exclusion principle, whereas the carriers of the fundamental forces must be bosons (that can be in the same single-particle states).

- Basis for explanations of:
  - lasers, Bose-Einstein condensates, superconductors, superfluids, the periodic table, neutron stars, the stability of matter, ...

1. Why SSC Needs an Explanation

- *A "profound impact" in non-relativistic quantum theories...*

"From the microscopic structure of atoms to the macroscopic structure of neutron stars, a dazzling wealth of physical phenomena would be incomprehensible without this spin–statistics rule. Many elements of condensed matter physics, for instance, band structure, Fermi liquid theory, superfluidity, superconductivity, quantum Hall effect, and so on and so forth, are consequences of this rule." (Zee 2010.)

The world would be a different place if spin-one-half particles were not subject to Pauli's exclusion principle. In all fundamental branches of modern (natural) science, the connection between particle spins and multiparticle behavior plays a crucial role, and to date, no physical system violating it has ever been observed. (Kuckert 2007, pg. 207.)

1. Why SSC Needs an Explanation

- *... whose explanation had to wait until relativistic quantum field theory (RQFT)...*

> "...the explanation of the spin–statistics connection by Fierz and by Pauli in the late 1930s, and by Luders and Zumino and by Burgoyne in the late 1950s, ranks as one of the great triumphs of relativistic quantum field theory." (Zee 2010.)

> "[The Spin-Statistics theorem]... clarifies one of the great mysteries of non-relativistic quantum theory: the contrasting symmetry properties of the wavefunctions of particles of integer (boson) versus half-integer (fermionic) spin." (Duncan 2012.)

1. Why SSC Needs an Explanation

- *... or did it?*

"The spin–statistics connection seems crucial to understanding the behavior of several physical systems for which relativistic considerations seem quite insignificant... Non-relativistic theories seem to adequately describe most of these systems and the spin–statistics connection has to be inserted 'by hand' when formulating these theories." (Shaji 2009.)

"An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity... we have not been able to find a way of reproducing his arguments on an elementary level. This probably means we do not have a complete understanding of the fundamental principle involved." (Feynman 1965.)

1. Why SSC Needs an Explanation

Spin–Statistics Theorem: Any physical system described by an RQFT must possess SSC.

(I) Does the Spin–Statistics Theorem explain SSC in RQFTs?

- *Them*: Yes.
- *Me*: No!

(II) What explains SSC in non-relativistic theories?

- *Them*: Big Mystery!
- *Me*: Not so much...

A Deductive–Nomological Explanation of SSC?

- DN explains by virtue of a derivation from a set of covering laws... (*etc.*).

- *Initial Problem*: SSC is a general regularity (principle, law).

> *Explanandum*: All physical systems of the relevant type possess SSC.

- *Problem of Conjunction*:
  (L_1 & L_2) entails L_1, but (L_1 & L_2) does not necessarily explain L_1.

A Deductive–Nomological Explanation of SSC?

- DN explains by virtue of a derivation from a set of covering laws... (*etc.*).

- *Initial Problem*: SSC is a general regularity (principle, law).

> *Explanandum*: All physical systems of the relevant type possess SSC.

- *Resolution*:
DN explains by virtue of a derivation from a set of *first principles*... (*etc.*).

> *Problem*: There is no unique set of first principles from which SSC can be derived in RQFTs.

2. The Spin–Statistics Theorem Does Not Explain SSC

Approach	Principles	Derived Property
Wightman (Luders & Zumino 1958; Burgoyne 1958)	(a) Lorentz invariance (b) Spectrum condition (c) NCR for fields	SSC for fields
Algebraic (Guido & Longo 1995)	(a) Modular covariance (b) Additivity (c) NCR for field observables (d) Algebraic causality	SSC for DHR "particle" representations
Lagrangian (Fierz 1939; Pauli 1940)	(a) Lorentz invariance (b) Spectrum condition (c) NCR for particles (d) Causality	SSC for fermionic fields
	(a) Lorentz invariance (b) NCR for particles (c) Causality	SSC for bosonic fields
Weinberg (Weinberg 1964)	(a) Lorentz invariance for S-matrix (b) Cluster decomp. for S-matrix (c) NCR for particles	SSC for particles

A Unifying Explanation of SSC?

- A unifying explanation explains by virtue of belonging to the most unifying systematization of the set K of claims currently endorsed by the scientific community.
- Let K be claims associated with RQFTs.

Problem: There is no consensus on how to systematize K (i.e., on which approach to RQFTs should be adopted).

2. The Spin–Statistics Theorem Does Not Explain SSC

Which Approach?

(I) Wightman/algebraic ("purist") approaches?

Problem of Empirical Import:
No *realistic interacting* models of the relevant axioms.

Qualifications

- *Realistic interacting* model = model for a 4-dim RQFT (e.g., QED, QCD) from which predictions have been derived and confirmed.
- *Non-interacting* models exist.
- *Unrealistic interacting* models exist: (Rivasseau 2003)
  - $P(\phi)_2$, ϕ^4_3, Yukawa model (2-dim, 3-dim), Gross-Neveu model (2-dim).

2. The Spin–Statistics Theorem Does Not Explain SSC

Which Approach?

(II) Lagrangian/Weinberg ("pragmatist") approaches?

Renormalization Problem: S-matrix assumes non-interacting multi-particle states at asymptotic times are related to interacting multi-particle states at finite times; and this requires introduction of infinitely renormalized parameters.

UV Problem: For typical *realistic interacting* QFTs, the power series expansion of the S-matrix contains divergent terms at high energies.

Convergence Problem: For typical *realistic interacting* QFTs, the power series expansion of the S-matrix may not converge.

2. The Spin–Statistics Theorem Does Not Explain SSC

<u>*Qualifications*</u>

- *Renormalization*, *UV*, *Convergence Problems* are common to any approach that employs renormalized perturbation theory to derive predictions from *realistic interacting* RQFTs.
- Some *realistic interacting* RQFTs (e.g., QCD) do not suffer the *UV Problem.*
- Renormalization Group techniques address *Renormalization* and *UV Problems.*

A Unifying Explanation of SSC?

- Does the Spin-Statistics theorem belong to an explanatory store $E(K)$ for $K =$ RQFT claims?

Kitcher (1989):
- K must be consistent and deductively closed.
- $E(K)$ must be unique.

Problem: No such K and $E(K)$ exists for RQFTs!

- Can a consistent K be identified?
- If so, will there be a unique $E(K)$?

A Causal Explanation of SSC?

- A causal explanation (of a "general" type of event/fact) explains by virtue of specifying possible causal histories.

> "A general causal explanation says what the causal histories of instances of the event-type being explained have in common, or says something about what it would have taken for a given alternative type of event to have occurred instead, which applies to many or most of the instances of the event-type being explained." (Skow 2013.)

- *Gloss*: A general causal explanation explains by virtue of placing constraints on *dynamically* possible states.
  - *Suppose dynamically possible trajectories supervene on causal histories.*

A Causal Explanation of SSC?

> *Problem*: The Spin–Statistics theorem explains (to the extent that it does explain) by virtue of placing a constraint on *kinematically* possible states.

- Any state of a physical system described by an RQFT must possess SSC, *regardless* of what dynamics it satisfies.

A Structural Explanation of SSC?

- The Spin–Statistics theorem demonstrates how a set of principles limits the kinematically possible states of physical systems to those that possess SSC.

"...a structural explanation can be understood as one in which the *explanandum* is explained by showing how the (typically mathematical) structure of the theory itself limits what sort of objects, properties, states, or behaviors are admissible within the frame-work of that theory, and then showing that the *explanandum* is in fact a consequence of that structure." (Bokulich 2011.)

A Structural Explanation of SSC?

Problems:

- Should the set of principles be taken to represent real physical structures? (Bueno & French 2012.)
  - *But*: Which structures? No unique set of principles.

- SSC essential to explanations in non-relativistic theories:
  - *Electronic structure of solids*
  - *Formation of white dwarf stars*
  - *Formation of superconductors and Bose–Einstein condensates*

- How does a structural explanation of SSC in RQFTs explain SSC in non-relativistic theories?

2. The Spin–Statistics Theorem Does Not Explain SSC

General Concern:

- Spin–Statistics theorem (*both* purist & pragmatist versions) demonstrates that SSC is an essential property of *non-interacting*, and at most *unrealistic interacting* RQFTs.
- In non-relativistic theories, SSC appears as a (brute fact) property of *interacting* theories:
  - *Electronic structure of solids*
  - *Formation of white dwarf stars*
  - *Formation of superconductors and Bose–Einstein condensates*
- If the explanandum is *SSC in realistic interacting theories*, then the Spin–Statistics theorem by itself does not provide an explanation.

3. What Would Explain SSC?

Weatherall's (2011) example.

- *Explanandum*: $m_i = m_g$ in Newontian gravity.
- *Relevant theories*: Newtonian gravity, GR.

> "The explanatory demand is to show how, given some superseding theory, a general fact as expressed within one theory is really necessary or to be expected within the regime in which the old theory is successful... The explanatory work, then, is done by presenting the details of the relationship between the two theories." (Weatherall 2011.)

Analogously...

- *Explanandum*: SSC in realistic interacting theories.
- *Relevant theories*: Non-relativistic quantum theories, RQFTs.

Continuing the Analogy:

General relativity	$\xrightarrow[\textit{flattening}]{\textit{lightcone}}$	Newton–Cartan gravity	$\xrightarrow[\textit{theorem}]{\textit{Trautman's}}$	Newtonian gravity
$m_i = m_g$ *is not expressible*		$m_i = m_g$ *is derivable*		$m_i = m_g$ *is a brute fact*

Realistic interacting RQFTs	$\xrightarrow[\textit{interaction}]{\textit{turn off}}$	Non-interacting RQFTs	$\xrightarrow{\textit{recovery}}$	Interacting non-relativistic theories
SSC is not expressible		*SSC is derivable*		*SSC is a brute fact*

- <u>*Now*</u>: Consider how recovery of *interacting* Galilei-invariant QFTs from *non-interacting* RQFTs can be performed.

3. What Would Explain SSC?

$$\begin{tikzcd}[column sep=large, row sep=huge]
\textbf{\textit{non-interacting}} \ \textbf{\textit{RQFTs}} \arrow[d, "\textit{intertheoretic relation}"] & \text{relativistic spin state} \arrow[r, leftrightarrow, "\mathit{SSC}", "\mathit{essential}"'] \arrow[d, "?"] & \text{relativistic statistics state} \arrow[d, "?"] \\
\textbf{\textit{interacting}} \ \textbf{\textit{GQFTs}} & \text{non-relativistic spin state} \arrow[r, leftrightarrow, dashed, "\mathit{SSC}", "\mathit{brute\ fact}"'] & \text{non-relativistic statistics state}
\end{tikzcd}$$

3. What Would Explain SSC?

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c}\boldsymbol{\textit{non-interacting}}\\ \boldsymbol{RQFTs}\end{array} \arrow[d, "\begin{array}{l}\textit{intertheoretic}\\ \textit{relation}\end{array}"] & \begin{array}{c}\text{representation}\\ \text{of } \mathcal{P}\end{array} \arrow[d, "?"] \arrow[r, leftrightarrow, "SSC", "\textit{essential}"'] & \begin{array}{c}\text{relativistic}\\ \text{statistics state}\end{array} \arrow[d, "?"] \\
\begin{array}{c}\boldsymbol{\textit{interacting}}\\ \boldsymbol{GQFTs}\end{array} & \begin{array}{c}\text{representation}\\ \text{of } \mathcal{G}\end{array} \arrow[r, leftrightarrow, dashed, "SSC", "\textit{brute fact}"'] & \begin{array}{c}\text{non-relativistic}\\ \text{statistics state}\end{array}
\end{tikzcd}
$$

3. What Would Explain SSC?

$$\begin{tikzcd}[column sep=large, row sep=large]
\textbf{\textit{non-interacting RQFTs}} \arrow[d, "\textit{intertheoretic relation}"] & \text{representation of } \mathcal{P} \arrow[r, leftrightarrow, "SSC", "\textit{essential}"'] \arrow[d, "\textit{speed–space contraction}"] & \text{relativistic NCR} \arrow[d, "\textit{speed–space contraction}"] \\
\textbf{\textit{interacting GQFTs}} & \text{representation of } \mathcal{G} \arrow[r, leftrightarrow, dashed, "SSC", "\textit{brute fact}"'] & \text{non-relativistic NCR}
\end{tikzcd}$$

<u>"Speed–space" contraction of $\mathcal{P}$ yields $\mathcal{G}$</u>: (Bacry & Lévy-Leblond 1968)

Replace $\mathbf{K}$, $\mathbf{P}$ with $\varepsilon\mathbf{K}$, $\varepsilon\mathbf{P}$, take limit $\varepsilon \to 0$ of Lie brackets.

- Limit of small velocities and small spatial translations.
- *Representations of $\mathcal{P}$ transform into representations of $\mathcal{G}$.*
- *Spacelike intervals transform into spatial intervals at equal times.*

3. What Would Explain SSC?

$$
\begin{tikzcd}
\textbf{\textit{non-interacting RQFTs}} \arrow[d, "\textit{intertheoretic relation}"] & \text{representation of } \mathcal{P} \arrow[d, "\textit{speed–space contraction}"] \arrow[r, leftrightarrow, "\textit{SSC}", "\textit{essential}"'] & \text{relativistic NCR} \arrow[d, "\textit{speed–space contraction}"] \\
\textbf{\textit{interacting GQFTs}} & \text{representation of } \mathcal{G} \arrow[r, leftrightarrow, dashed, "\textit{SSC}", "\textit{brute fact}"'] & \text{non-relativistic NCR}
\end{tikzcd}
$$

- A *kinematic* result: Doesn't map relativistic dynamics to non-relativistic dynamics. (Brown & Holland 2003)
  - *But*: SSC is a purely kinematical property.
  - *And:* There are representations of $\mathcal{G}$ that describe *realistic interacting* GQFTs.

3. What Would Explain SSC?

$$\begin{tikzcd}
\textbf{\textit{non-interacting RQFTs}} \arrow[d, "\textit{intertheoretic relation}"] & \text{representation of } \mathcal{P} \arrow[r, leftrightarrow, "\textit{SSC}", "\textit{essential}"'] \arrow[d, "\textit{speed–space contraction}"] & \text{relativistic NCR} \arrow[d, "\textit{speed–space contraction}"] \\
\textbf{\textit{interacting GQFTs}} & \text{representation of } \mathcal{G} \arrow[r, leftrightarrow, dashed, "\textit{SSC}", "\textit{brute fact}"'] & \text{non-relativistic NCR}
\end{tikzcd}$$

- Doesn't explain SSC in realistic interacting RQFTs.
  - No intertheoretic relation between non-interacting RQFTs and realistic interacting RQFTs.

4. Conclusion

- The Spin–Statistics theorem does not explain SSC.
- The Spin–Statistics theorem coupled with an appropriate intertheoretic relation between *non-interacting* RQFTs and *interacting* non-relativistic quantum theories explains SSC in the latter.
- There is currently no adequate explanation of SSC in realistic interacting RQFTs.

References

Barcy, H. & J-M. Lévy-Leblond (1968) 'Possible Kinematics', *J. Math. Phys. 9*, 1605.

Bokulich, A. (2011) 'How Scientific Models Can Explain', *Synthese 180*, 33.

Brown, H. & P. Holland (2003) 'The Non-Rel Limits of the Maxwell and Dirac Equations', *SHPMP 34*, 161.

Bueno, O. & S. French (2012) 'Can Mathematics Explain Physical Phenomena?', *BJPS 63*, 85.

Burgoyne, N. (1958) 'On the Connection of Spin and Statistics', *Nuovo Cimento 8*, 607.

Duncan, A. (2012) *The Conceptual Framework of Quantum Field Theory*, Oxford Univ. Press.

Fierz, M. (1939) 'Über die Relativistische Theorie...', *Hel. Phys. Acta, 12*, 3.

Feynman, R. (1965) *The Feynman Lectures on Physics, Vol. 3*, Addison-Wesley.

Guido, D. & R. Longo (1995) 'An Algebraic Spin and Statistics Theorem', *Comm. Math. Phys. 172*, 517.

Kitcher, P. (1989) 'Explanatory Unification and the Causal Structure of the World', in Kitcher & Salmon (eds.) *Scientific Explanation*, Univ. Minnesota Press, 410.

Kuckert, B. (2007) 'The Classical and Quantum Roots of Pauli's Spin-Statistics Relation', in W. Unruh & R. Schützhold (eds.) *Quantum Analogues*, Springer, 207.

Lüders, G. & B. Zumino (1958) 'Connection Between Spin and Statistics', *Phys. Rev. 110*, 1450.

Pauli, W. (1940) 'The Connection Between Spin and Statistics', *Phys. Rev. 58*, 716.

Rivasseau, V. (2003) 'An Introduction to Renormalization', *Poincaré Seminar 2002*, 139.

Shaji, A. (2009) 'Sudarshan's Non-Rel Approach to the Spin-Stats Theorem', *J Phys Conf Series 196*, 012013.

Skow, B. (2013) 'Are There Non-Causal Explanations (of Particular Events)?', *BJPS 65*, 445.

Weatherall, J. (2011) 'On (Some) Explanations in Physics', *Phil. Sci. 78*, 421.

Weinberg, S. (1964) 'Feynman Rules for Any Spin', *Physical Review 133*, B1318.

Werner, S. *et al.* (1975) 'Observation of the Phase Shift of a Neutron Due to Precession in a Magnetic Field', *Phy Rev Lett 35*, 1053.

Zee, A. (2010) *Quantum Field Theory in a Nutshell*, Princeton Univ. Press.