Pragmatists and Purists on CPT Invariance in Relativistic Quantum Field Theories

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1. Pragmatism, Purity, and Foundations.
2. Pragmatism *vs*. Purity on CPT Invariance.
3. Greenberg on Relativity and CPT Invariance.
4. Causal Perturbation Theory.
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1. Pragmatism, Purity, and Foundations.

What version of QFT should inform philosophical investigations of foundational issues?

CQFT (cutoff/conventional QFT): (Wallace 2011)

- CQFT resolves problems associated with renormalized perturbation theory.
- CQFT is empirically successful.

AQFT (axiomatic/algebraic QFT): (Fraser 2011)

- Renormalization techniques reflect the empirical content of QFT; thus are not the exclusive property of CQFT.
- AQFT offers a rigorous foundation for the theoretical content of QFT.

1. Pragmatism, Purity, and Foundations.

What version of QFT should inform philosophical investigations of foundational issues?

Pragmatist approaches:

- Directly engage with renormalized perturbation theory, and thus the *empirical content* of QFT.
- Face problems with mathematical rigor with respect to the *theoretical content* of QFT.

Purist approaches:

- Do not engage with renormalized perturbation theory, thus do not engage with the *empirical content* of QFT.
- Seek to place *theoretical content* of QFT on rigorous mathematical foundations.

2. Pragmatism versus Purity on CPT Invariance.

CPT invariance:
Invariance under charge conjugation (C), space inversion (P), and time reversal (T).

- *Task*: To consider concrete examples of purist and pragmatist formulations of the CPT theorem:

  (a) *Purist example:* Wightman axiomatic CPT theorem.

  (b) *Pragmatist example:* Weinberg's CPT theorem.

- *Immediate goal*: To identify explicit problems associated with these formulations that will make the distinction between pragmatism and purity more concrete.

2. Pragmatism versus Purity on CPT Invariance.

(a) The Wightman Axiomatic Approach

- Basic objects are vacuum expectation values of (unordered) products of fields referred to as Wightman functions...

$$W^{(n)}(f_1, \ldots, f_n) \equiv \langle 0|\phi(f_1)\ldots\phi(f_n)|0\rangle$$

- ...that satisfy the Wightman axioms.

- Fields are defined as operator-valued distributions:

$$\phi(f) = \int \phi(x)f(x)dx$$

$f \in \mathcal{D}(\mathbb{R}^4)$, space of continuous test functions with compact support.

$\phi \in \mathcal{D}'(\mathbb{R}^4)$, dual space of continuous linear functionals on $\mathcal{D}(\mathbb{R}^4)$.

2. Pragmatism versus Purity on CPT Invariance.

(a) The Wightman Axiomatic Approach

(i) *Restricted Lorentz Invariance* (RLI): The fields are invariant under the restricted Lorentz group.

(ii) *Spectrum Condition* (SC): The fields possess positive energy.

(iii) *Weak Local Commutativity* (WLC):

$$\langle 0|\phi(f_1)\ldots\phi(f_n)|0\rangle = i^K\langle 0|\phi(f_n)\ldots\phi(f_1)|0\rangle.$$

Axiomatic CPT Theorem: (Jost 1957)

(RLI & SC & WLC) $\Rightarrow$ (CPT of fields)

2. Pragmatism versus Purity on CPT Invariance.

(a) The Wightman Axiomatic Approach

Problem of Empirical Import:
No non-trivial interacting models of the Wightman axioms.

- *Suggests:* *CPT invariance is restricted to free (or trivially interacting) states.*

UV Problem:
The product of distributions at the same point is not in general well-defined.

- *Solution for free fields*: *Normal-ordering.*

2. Pragmatism versus Purity on CPT Invariance.

(b) Weinberg's Approach

- The basic object is the S-matrix, encoding probability amplitudes for particle scattering events.

(i) *Perturbation Theory*: The S-matrix is given by a perturbative expansion in time-ordered products of $\mathfrak{H}_{int}(x)$:

$$S_{\beta\alpha} = \sum_{n=0}^{\infty} \frac{-i^n}{n!} \int_{-\infty}^{\infty} \langle\beta| T\{\mathfrak{H}_{int}(x_1)\ldots\mathfrak{H}_{int}(x_n)\}|\alpha\rangle d^4x_1\ldots d^4x_n$$

(ii) *RLI of S-matrix*: The S-matrix is invariant under restricted Lorentz transformations.

(iii) *Cluster Decomposition* (CD): Let $S_{\beta_1+\ldots+\beta_N,\ \alpha_1+\ldots+\alpha_N}$ represent the S-matrix for N multi-particle processes $|\alpha_1\rangle\rightarrow|\beta_1\rangle,\ldots,|\alpha_N\rangle\rightarrow|\beta_N\rangle$. If all particles in states $|\alpha_i\rangle$, $|\beta_i\rangle$ are spacelike separated from all particles in states $|\alpha_j\rangle$, $|\beta_j\rangle$, $i \neq j$, then the S-matrix factorizes:

$$S_{\beta_1+\ldots+\beta_N,\ \alpha_1+\ldots+\alpha_N} = S_{\beta_1\alpha_1}\ldots S_{\beta_N\alpha_N}$$

2. Pragmatism versus Purity on CPT Invariance.

(b) Weinberg's Approach

- *Result* 1. A sufficient condition for the compatibility of RLI and CD is that $\mathfrak{H}_{int}(x)$ is a functional of free fields that satisfy RLI and local commutativity.

- *Result* 2. If the fields possess a conserved charge, then anti-particle states must be posited.

- *Result* 3. The full Hamiltonian density $\mathfrak{H}(x)$ is CPT invariant.

Weinberg's CPT Theorem: (Weinberg 1995)

[(RLI of S-matrix) & CD & (existence of conserved charges)]
$\Rightarrow$ (CPT invariance of $\mathfrak{H}(x)$)

2. Pragmatism versus Purity on CPT Invariance.

(b) Weinberg's Approach

Renormalization Problem:
To guarantee the existence of a non-trivial S-matrix, interacting fields and free asymptotic fields are related by infinitely renormalized parameters.

UV Problem: The power series expansion of the S-matrix contains divergent terms at high energies/short distances.

- *Deeper issue*: *Product of interacting fields at the same point is not well-defined.*

Existence Problem: The power series expansion of the S-matrix may not exist as a convergent series.

2. Pragmatism versus Purity on CPT Invariance.

(b) Weinberg's Approach

- The Renormalization Problem and the UV Problem are conceptually distinct:

> "...[T]he renormalization of masses and fields has nothing directly to do with the presence of infinities, and would be necessary even in a theory in which all momentum space integrals were convergent." (Weinberg 1995, pg. 441.)

> "Concerning the relation between renormalization and the removal of UV divergences it must be stressed that these are at first hand quite different problems. Renormalization... is necessary independent of the occurrence of UV divergences, if we want to describe the theory in terms of directly measureable parameters..." (Steinmann 2000, pg. 83.)

3. Greenberg on Relativity and CPT Invariance.

Claim (Greenberg 2002)
If CPT invariance is violated in an interacting RQFT, then so is Lorentz invariance.

- *Suggests*:
  - Tests for violations of Lorentz invariance *via* experiments that measure CPT violation. (Kostelecky 2011)
  - A mysterious connection between a spacetime symmetry and a discrete symmetry. (Greaves 2010; Arntzenius & Greaves 2009)

- *But*: Greenberg's proof is formulated in the axiomatic approach; and it fails to address the *Problem of Empirical Import* and the *UV Problem*.
- *Thus*: It cannot be interpreted as a claim about interacting RQFTs. (Dütsch & Gracia-Bondía 2012)

3. Greenberg on Relativity and CPT Invariance.

Greenberg's time-ordered Wightman function:

$$\tau^{(n)}(x_1,\ldots,x_n) \equiv \sum_p \theta(t_{p_1} - t_{p_2})\ldots\theta(t_{p_{n-1}} - t_{p_n})W^{(n)}(x_{p_1},\ldots,x_{p_n})$$

Compare with pragmatist τ-function:

$$\tau^{(n)}(x_1,\ldots,x_n) \equiv \sum_p \theta(t_{p_1} - t_{p_2})\ldots\theta(t_{p_{n-1}} - t_{p_n})\langle 0|\phi(x_{p_1})\ldots\phi(x_{p_n})|0\rangle$$

Example: Feynmann propagator:

$$\begin{aligned}\tau^{(2)}(x_1, x_2) &= \theta(t_1 - t_2)\langle 0|\phi(x_1)\phi(x_2)|0\rangle + \theta(t_2 - t_1)\langle 0|\phi(x_2)\phi(x_1)|0\rangle \\ &\equiv \Delta_F(x_1, x_2)\end{aligned}$$

Significance of τ-functions:

S-matrix elements in pragmatist approaches can be reduced to expressions involving interacting $\tau^{(n)}$s.

3. Greenberg on Relativity and CPT Invariance.

Greenberg's time-ordered Wightman function:

$$\tau^{(n)}(x_1,\ldots,x_n) \equiv \sum_p \theta(t_{p_1} - t_{p_2})\ldots\theta(t_{p_{n-1}} - t_{p_n})W^{(n)}(x_{p_1},\ldots,x_{p_n})$$

- *Greenberg shows:* If $\tau^{(n)}$ is RLI, then $W^{(n)}$ satisfies *Weak Local Commutativity* (WLC).
- *Thus*: (RLI of $\tau^{(n)}$) $\Rightarrow$ (WLC of $W^{(n)}$)

$$\Rightarrow \begin{pmatrix} \text{CPT invariance of } W^{(n)} \\ \text{that satisfy RLI and SC} \end{pmatrix}$$

So: If Lorentz invariance of an interacting RQFT requires RLI of its τ-functions, then

"... if CPT invariance is violated in an interacting quantum field theory, then that theory also violates Lorentz invariance."

3. Greenberg on Relativity and CPT Invariance.

Greenberg's time-ordered Wightman function:

$$\tau^{(n)}(x_1,\ldots,x_n) \equiv \sum_p \theta(t_{p_1}-t_{p_2})\ldots\theta(t_{p_{n-1}}-t_{p_n})W^{(n)}(x_{p_1},\ldots,x_{p_n})$$

A purist claim about CPT invariance of interacting states?

Problem of Empirical Import: Non-trivial interacting Wightman functions do not exist.

UV Problem: Products of Heaviside functions and Wightman functions are not well-defined.

- $\tau^{(n)}$ only exists as a distribution in $\mathcal{D}'(\mathbb{R}^{4n}\backslash\{0\})$.

3. Greenberg on Relativity and CPT Invariance.

Greenberg's time-ordered Wightman function:

$$\tau^{(n)}(x_1,\ldots,x_n) \equiv \sum_p \theta(t_{p_1}-t_{p_2})\ldots\theta(t_{p_{n-1}}-t_{p_n})W^{(n)}(x_{p_1},\ldots,x_{p_n})$$

A pragmatist claim about CPT invariance?

- Interacting τ-functions exist (on pain of confronting the *Renormalization Problem*).
- *But*: CPT invariance follows RLI only under Jost's axiomatic proof of the CPT Theorem (which assumes SC).

4. Causal Perturbation Theory.

- *Locus classicus*: Epstein & Glaser (1973).
- *Offers*:
  - A regularization scheme as response to *UV Problem*.
  - An axiomatic scheme as response to *Renormalization Problem*.
  - A version of the CPT theorem. (Dütsch & Gracia-Bondía 2012)
- Mediation in debate between purists and pragmatists?

4. Causal Perturbation Theory.

- The basic object is the S-matrix (*à la* Weinberg), conceived as a formal power series in operator-valued distributions (*à la* Wightman).

(i) *Formal S-matrix*: The S-matrix is given by a formal power series in operator-valued distributions $S_n(g)$, $g \in \mathcal{D}(\mathbb{R}^4)$.

$$S(g) = \sum_{n=0}^{\infty} \frac{-i^n}{n!} \int_{-\infty}^{\infty} S_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n) d^4 x_1 \ldots d^4 x_n$$

(ii)-(iv) *Translation invariance*, *Lorentz invariance*, *Unitarity*.

(v) *Causality*:

$S(g_1 + g_2) = S(g_1) S(g_2)$, if $\mathrm{supp}(g_1)$ is spacelike separated from $\mathrm{supp}(g_2)$.

4. Causal Perturbation Theory.

Main result. (Epstein & Glaser 1973)

Assume $S_0 = 1$ and $S_1 = \mathfrak{H}_{int}(x)$. Then for $n \geq 2$,

(i) S_n consist of time-ordered products of fields that appear in $\mathfrak{H}_{int}(x)$.

(ii) $S_n \in \mathcal{D}'(\mathbb{R}^{4n}\backslash\{0\})$.

- *Recall Greenberg's purist UV Problem*: $\boldsymbol{\tau}^{(n)}$ only exists in $\mathcal{D}'(\mathbb{R}^{4n}\backslash\{0\})$.
- *Task:* To extend a distribution defined in $\mathcal{D}'(\mathbb{R}^{4n}\backslash\{0\})$ to one defined in $\mathcal{D}'(\mathbb{R}^{4n})$.

4. Causal Perturbation Theory.

Definition : (Steinmann 1971)

The *scaling degree* ω of $t \in \mathcal{D}'(\mathbb{R}^n)$ *w.r.t.* the origin is the greatest lower bound of $\{\omega' \in \mathbb{R} : \lim_{\lambda \to 0} \lambda^{\omega'} t(\lambda x) = 0\}$.

- *Singular order* ρ (*superficial degree of divergence*): $\rho = \omega - n$.

Theorem: (Brunetti & Fredenhagen 2000)

Let $t_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ have scaling degree ω *w.r.t.* the origin.

(i) If $\rho < 0$, there exists a unique $t \in \mathcal{D}'(\mathbb{R}^n)$ with scaling degree ω such that $t(f) = t_0(f)$, $\forall f \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$.

(ii) If $\rho \geq 0$, there exists a $t \in \mathcal{D}'(\mathbb{R}^n)$ with scaling degree ω such that $t(f) = t_0(f)$, $\forall f \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$, and which is uniquely determined by its values on a finite set of test functions.

4. Causal Perturbation Theory.

- For case (ii), any $f \in \mathcal{D}(\mathbb{R}^n)$ can be decomposed as:

$$f(x) = f_1(x) + f_2(x)$$

$$= \underbrace{\sum_{|\alpha|\leq\rho} (\partial^\alpha f)(0) w_\alpha(x)}_{\text{Weighted Taylor expansion of } f \text{ at origin up to order } \rho.} + \underbrace{\sum_{|\alpha|=|\rho|+1} x^\alpha \psi_\alpha(x)}_{\text{Function that vanishes up to order } \rho \text{ at origin.}}$$

$$\boxed{\begin{gathered} w_\alpha,\ \psi_\alpha \in \mathcal{D}(\mathbb{R}^n) \\ \partial^\alpha w_\beta(0) = \delta^\alpha_\beta \end{gathered}}$$

Function that vanishes up to order ρ at origin. $=$ *Function in $\mathcal{D}(\mathbb{R}^n \backslash \{0\})$.*

- Fact: $x^\alpha t_0 \in \mathcal{D}'(\mathbb{R}^n \backslash \{0\})$ has singular order < 0. Thus it has a unique extension, call it t'.
- An extension t of t_0 can then be given by:

  (a) $t(f_2) \equiv t'(f_2)$.

  (b) $t(f_1)$ is determined by $t(w_\alpha) = C_\alpha$ for arbitrary constants C_α.
- Thus: $t = t' + \sum_{|\alpha|\leq\rho} C_\alpha (\partial^\alpha \delta)$

4. Causal Perturbation Theory.

- ***Thus:*** Can use method of distribution extension to obtain all terms in $S(g)$ as uniquely defined distributions with corresponding Hilbert space operators.
- A purist solution to the *UV Problem*.
- ***Moreover***: If CPT invariance holds prior to regularization *via* distribution extension, it holds after regularization *via* distribution extension:

Causal Pert. Theory CPT Theorem: (Dütsch & Gracia-Bondía 2012)

[(Causal pert. axioms) & (CPT inv. of free fields and interaction)]

$\Rightarrow$ (CPT invariance of regularized time-ordered products)

4. Causal Perturbation Theory.

- *Obstacle #1*: To interpret $S(g)$ as physical S-matrix, need to get rid of test functions $g(x)$: Take "adiabatic limit" $g(x) \to 1$.
  - *But*: Only exists (in strong sense) for massive theories.

- *Obstacle #2*: Does the resulting formal expression $S(g)$ exist as a convergent series for non-trivial interacting RQFTs?

> *Existence Problem*: The power series expansion of the S-matrix may not exist as a convergent series.

- This *pragmatist* problem, in causal perturbation theory, is just the purist's *Problem of Empirical Import*:

> *Problem of Empirical Import*:
> No non-trivial interacting models of causal pert. axioms.

5. Conclusion.

<u>*CQFT (cutoff/conventional QFT):*</u> (Wallace 2011)

- *Renormalization Problem:* Solved by re-defining renormalized parameters as functions of a UV cut-off.
- *UV Problem:* Solved by inserting UV cut-off.
- *Existence Problem:* Unsolved.

<u>*AQFT (axiomatic/algebraic QFT):*</u> (Fraser 2011)

- *Problem of Empirical Import:* Unsolved.
- *UV Problem:* Unsolved.

<u>*Causal Perturbation Theory*</u>:

- *UV Problem:* Solved by method of extension of distributions.
- *Problem of Empirical Import* converted to *Existence Problem*: Unsolved.

5. Conclusion.

Two Lessons:

1. Problems associated with renormalization should be made distinct from problems associated with perturbation theory.
   - *Purists should acknowledge the success of the use of perturbation theory in pragmatist approaches: One can object to renormalization as mathematically suspect while engaging rigorously with perturbation theory.*

2. The *Problem of Empirical Import* is not necessarily conceptually distinct from the *Existence Problem*.
   - *Pragmatists should acknowledge that purists may be taken to task on the Problem of Empirical Import to the same extent that pragmatists may be taken to task on the Existence Problem.*

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