Category-Theoretic Structure and Radical Ontic Structural Realism

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1. No Relations Without *Relata*?
2. An Analogy From General Relativity.
3. How To Do Category-Theoretic Physics.

1. No Relations Without *Relata*?

Radical Ontic Structural Realism (French & Ladyman 2003)

Structure consists of relations devoid of *relata*.

> "...when it comes to the physical world, the point at issue are concrete relations that are instantiated in the physical world and that hence are particulars in contrast to universals. For the relations to be instantiated, there has to be something that instantiates them... ."
>
> (Esfeld and Lam 2008)

> "As applied to a particular relation, this assertion seems incoherent. It only makes sense if it is interpreted as the metaphysical claim that ultimately there are only relations; that is, in any given relation, all of its *relata* can in turn be interpreted as *relations*."
>
> (Stachel 2006)

1. No Relations Without *Relata*?

Radical Ontic Structural Realism (French & Ladyman 2003)

Structure consists of relations devoid of *relata*.

> "Taken at face value... [radical ontic structural realism] is clearly incoherent..."
>
> (Wüthrich 2008)

> "I daresay that no ontic structural realist should be falling into the trap of accepting the view that 'relations can exist without *relata'*."
>
> (Dorato 2008)

1. No Relations Without *Relata*?

Radical Ontic Structural Realism (French & Ladyman 2003)

Structure consists of relations devoid of *relata*.

Untenable?

Set-theoretically, perhaps so.

- Suppose *structure* = *isomorphism class of structured sets* = $[\{X, R_i\}]$.
- A (*binary*) *relation* R on X is a subset of $X \times X$, the set of all ordered pairs (x_1, x_2), $x_1, x_2 \in X$.
- An *ordered pair* (x_1, x_2) is the set $\{x_1, \{x_1, x_2\}\}$.
- Ineliminable reference to elements ("*relata*") of a set.

1. No Relations Without *Relata*?

Radical Ontic Structural Realism (French & Ladyman 2003)

Structure consists of relations devoid of *relata*.

Untenable?

Category-theoretically, perhaps not.

- Suppose *structure* = *object in a category*.
- Primitives: objects, morphisms between objects.
- Reference to "internal constituents" of an object ("elements") can only be done in terms of other "external" objects and morphisms.

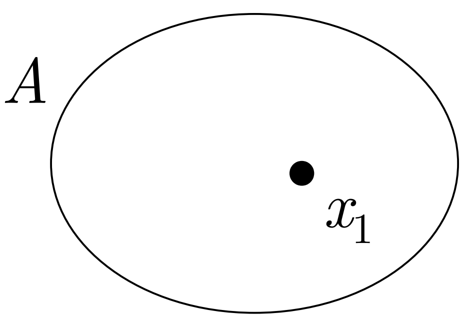
1. No Relations Without *Relata*?

- An ***element*** of an object A in a category $\mathbf{C}$ is a morphism $\mathbf{1} \to A$, where $\mathbf{1}$ is the *terminal object* in $\mathbf{C}$.

> An object $\mathbf{1}$ of a category $\mathbf{C}$ is a *terminal object* of $\mathbf{C}$ if for each object X of $\mathbf{C}$, there is exactly one $\mathbf{C}$-morphism $X \to \mathbf{1}$.

Set Theory

Primitives: sets, $\in$



$x_1 \in A$

Category Theory

Primitives: objects, morphisms

$$\mathbf{1} \xrightarrow{x_1} A$$

1. No Relations Without *Relata*?

- The *Cartesian product* of an object X with itself is an object P, together with a pair of morphisms $p_1 : P \to X$, $p_2 : P \to X$ such that, for any arbitrary object T with morphisms $f_1 : T \to X$, $f_2 : T \to X$, there is exactly one morphism $f : T \to P$ for which $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.

$$
\begin{tikzcd}
 & & X \\
T \arrow[r, "f"] \arrow[urr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & P \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\
 & & X
\end{tikzcd}
$$

- External probe (T, f_1, f_2, f) encodes internal pair structure of P.

1. No Relations Without *Relata*?

Objection: Elimination of relata in name only.

- Where set theory sees "elements", category theory sees "morphisms from the terminal object".
- "*No relations without relata*" becomes "*No objects (of the relevant type) without morphisms from the terminal object*".

Response

- Set-theoretic *relata* have correlates in category theory, but these correlates are not essential to the articulation of the relevant structure.
  - Category-theoretic objects need not be structured sets.
  - The structure encoded in such objects does not depend in an essential way on their elements.
  - Such objects have roles to play in articulating relevant notions of structure in physics.

2. An Analogy from General Relativity

General Relativity

Tensor formalism $\quad\quad$ **Einstein algebra formalism**

$$(M,\ g_{ab}) \xleftrightarrow{\text{1-1}} (C,\ g)$$

differentiable manifold

metric field satisfying Einstein equations

points of M correspond to maximal ideals of C

commutative ring

multilinear map on space of derivations of C and its dual, satisfying Einstein equations

- *Idea*: Reconstruct M as collection of maximal ideals of commutative ring $C^\infty(M)$ of smooth functions on M.
- *Different Indivs.-based Ontologies*: points *vs.* ideals
- *Common Structure:* Differentiable structure
- ***Elimination of points in name only?***

2. An Analogy from General Relativity

Consider: GR with asymptotic boundary conditions.

Tensor Models

- Replace M with *manifold with boundary* $M' = M \bigcup \partial M$.
- $(M,\ g_{ab})$ is Diff(M)-invariant.
- $(M',\ g_{ab})$ is Diff$_c$(M)-inv., but *not necessarily* Diff(M)-inv.

diffeomorphisms on M with compact support $\approx$ *"local" diffeomorphisms*

- No morphisms that preserve *both* M and M'.
- M and M' belong to *different* categories.

2. An Analogy from General Relativity

Consider: GR with asymptotic boundary conditions.

Einstein Algebra Models

- Replace *ring* $C \cong C^\infty(M)$ with *sheaf* $\mathscr{C} \cong C^\infty(M')$.
- Replace *Einstein algebra* (C, g) with *sheaf of Einstein algebras* $(\mathscr{C}, g)$.
    - $(\mathscr{C}, g)$ does not necessarily have *global cross sections* (*i.e.*, "elements").
    - (C, g) and $(\mathscr{C}, g)$ are objects in a *single category*: the category of "Einstein structured spaces". (Heller and Sasin 1995)

2. An Analogy from General Relativity

Upshot:

- Structure of tensor models: *"local" differentiable structure.*
  - Predicated directly on points of M.

- Structure of EA models: *"global" differentiable structure.*
  - Encoded directly in a sheaf of Einstein algebras.
  - *Not* predicated on maximal ideals of a single Einstein algebra.

- Tensor models (M, g_{ab}) are structured sets.
- Einstein structured spaces $(\mathscr{C}, g)$ are not!

2. An Analogy from General Relativity

Thus:

(1) The point correlates (maximal ideals) in Einstein algebra models of GR do not play an essential role in articulating the relevant notion of structure.

(2) Einstein algebra models of GR provide a more unifying description of phenomena in GR.

Analogously:

(1′) The correlates of set-theoretic *relata* in category theory do not play an essential role in articulating the relevant notion of structure.

(2′) This notion of structure does actual work in providing a more unifying description of phenomena.

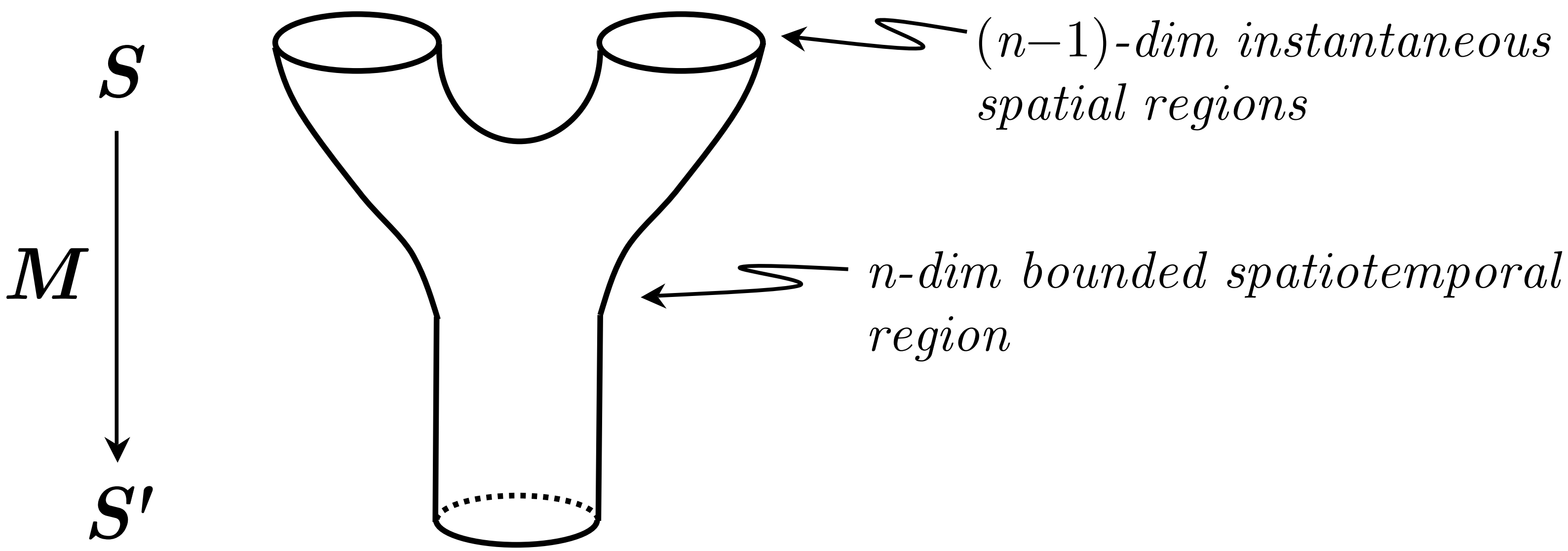
3. How To Do Category-Theoretic Physics

- Two more examples of categories, ***n*Cob** and **Hilb**. (Baez 2006)
    - Objects are not structured sets.
    - *Thus*: The *elements* of these objects, while well-defined, do not play an essential role in articulating the relevant notions of structure.
    - *Moreover*: These notions of structure are relevant to the pursuit of unifying descriptions of physical phenomena.

3. How To Do Category-Theoretic Physics

Ex. 1: The category n**Cob**.

- Objects: $(n-1)$-dim topological manifolds.
- Morphisms: n-dim top. manifolds with boundary (cobordisms).



3. How To Do Category-Theoretic Physics

Set-theoretically:

- Topological spaces are structured sets.
- Functions that preserve their structure are homeomorphisms.

Category-theoretically:

- Objects of $\boldsymbol{n}$**Cob** are not structured sets: morphisms are not even functions.
- Unlike **Set**, $\boldsymbol{n}$**Cob** admits a tensor product but no Cartesian product, and a *-morphism ($\boldsymbol{n}$**Cob** is a *monoidal *-category*).
- Elements of $\boldsymbol{n}$**Cob** objects are manifold points, but such elements do not play an essential role in articulating the structure of these objects.

Ex. 2: The category **Hilb**.

- Objects: finite-dim Hilbert spaces.
- Morphisms: bounded linear operators.

Set-theoretically:

- Hilbert spaces are structured sets.
- Functions that preserve their structure are unitary operators.

Category-theoretically:

- Objects of **Hilb** are not structured sets: general bounded linear operators need not preserve inner product.
- Unlike **Set**, **Hilb** is a monoidal *-category.
- Elements of **Hilb** objects are vectors, but vectors do not play an essential role in articulating the structure of these objects.

3. How To Do Category-Theoretic Physics

Claim: $\boldsymbol{n}$**Cob** and **Hilb** are relevant to the pursuit of unifying descriptions of physical phenomena. (Atiyah 1989)

- A *topological quantum field theory* (TQFT) is a functor

$$Z : \boldsymbol{n}\mathbf{Cob} \to \mathbf{Hilb}$$

- To every $(n{-}1)$-dim manifold S, Z assigns a Hilbert space $Z(S)$.
- To every n-dim cobordism $M : S \to S'$, Z assigns a linear operator $Z(M) : Z(S) \to Z(S')$.
- $Z(M'M) = Z(M')Z(M)$, for any n-dim corbodisms M, M'.
- $Z(1_S) = 1_{Z(S)}$, for any $(n{-}1)$-dim manifold S.

4. Conclusion.

- A category-theoretic notion of structure does not depend essentially on (set-theoretic) *relata*.
- Category-theoretic notions of structure have non-trivial physical applications.

<u>What the Category-theoretic Radical Ontic Structural Realist must do:</u>

- Provide rationale for fundamentality of category theory over set theory. (Pedroso 2009)
- Provide additional category-theoretic formulations of scientific theories that do not presuppose **Set**. (Döring & Isham 2008; Isham and Butterfield 2000; Baez 2006)

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