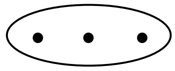
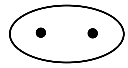
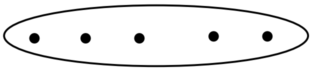
17: Sums

Topics
1. Sums of Objects
2. Sums Involving Terminal Objects
3. Negative Objects

1. Sums of Objects

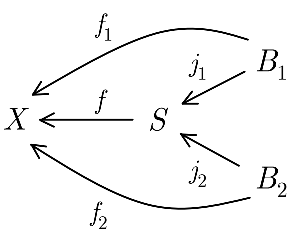
<u>*Consider*</u>: What does it mean to add 3 and 2, when 3 and 2 are represented as sets?

Take the elements of the sets  and  and "inject" them into a new set S

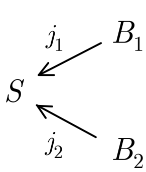
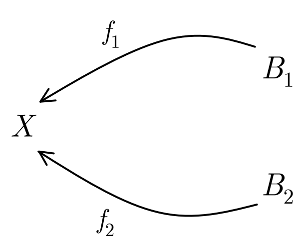
 that has as elements *all and only all* elements of the first two.

- How do we guarantee that S has *all and only all* elements of the first two in the "external" talk of category theory?
- Require that any map from S to any other object X preserves the "injections".

> <u>***Definition:***</u> A ***sum*** of two objects B_1, B_2 is an object S *together* with a pair of maps $j_1: B_1 \to S$, $j_2: B_2 \to S$, such that, for any object X with maps $f_1: B_1 \to X$, $f_2: B_2 \to X$, there is *exactly one* map $f: S \to X$ such that $f_1 = f \circ j_1$ and $f_2 = f \circ j_2$.

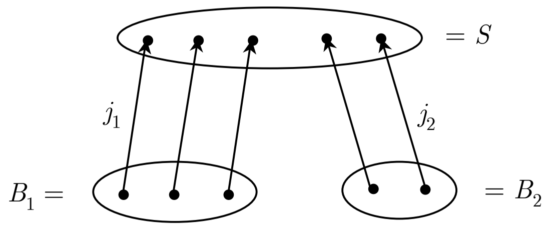


Call S, "$B_1 + B_2$"
j_1 and j_2 are the "sum injections".
f is required to do the same thing to the elements of S that f_1, f_2 do.
<u>*So*</u>: f takes the "B_1-elements" of S to X *separately* from the "B_2-elements".

<u>***So:***</u>  is a sum just when, for every 

there is exactly one $X \xleftarrow{f} S$ that makes the pieces "fit together" (*i.e.*, "commute").

Example 1: A sum in $\boldsymbol{\mathcal{S}}$.



Claim: This defines a sum: For *any* other set X with maps $f_1: B_1 \to X$, $f_2: B_2 \to X$, there is *exactly one* map $f: S \to X$ such that $f_1 = f \circ j_1$ and $f = f_2 \circ j_2$.

Proof: Suppose X is any set with maps $f_1: B_1 \to X$, $f_2: B_2 \to X$. Now define a map $f: S \to X$ in the following way;

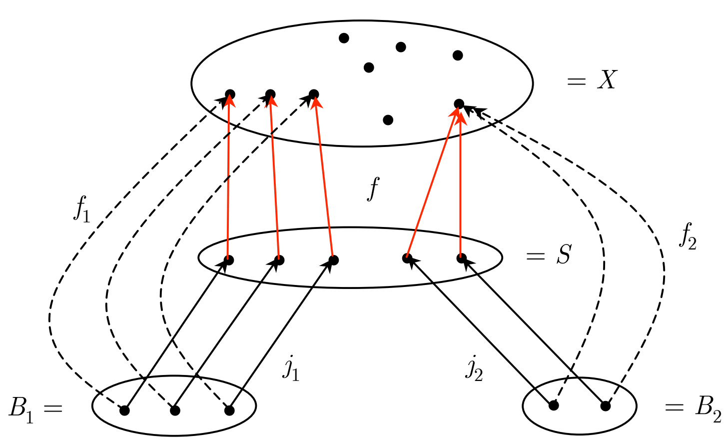
$$f(s) = \begin{cases} f_1(s') & \text{if } s = j_1(s') \\ f_2(s'') & \text{if } s = j_2(s'') \end{cases}$$

This guarantees that $f_1(s) = f(j_1(s))$ (or $f_1 = f \circ j_1$)
$f_2(s) = f(j_2(s))$ (or $f_2 = f \circ j_2$)

Is there only *one* such f?

Yes! Because
1. j_1, j_2 are *injective* (every dot in domain gets only one dot in codomain).
2. j_1, j_2 *together* are exhaustive of S (no dots in S left over).
3. j_1, j_2 don't overlap in S (no dots in S common to both of them).

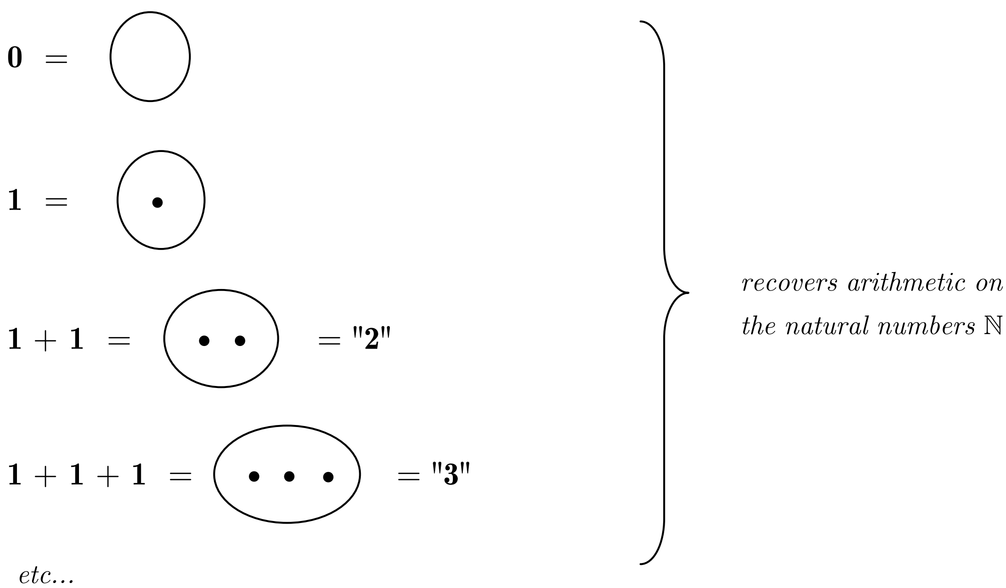
Particular case:



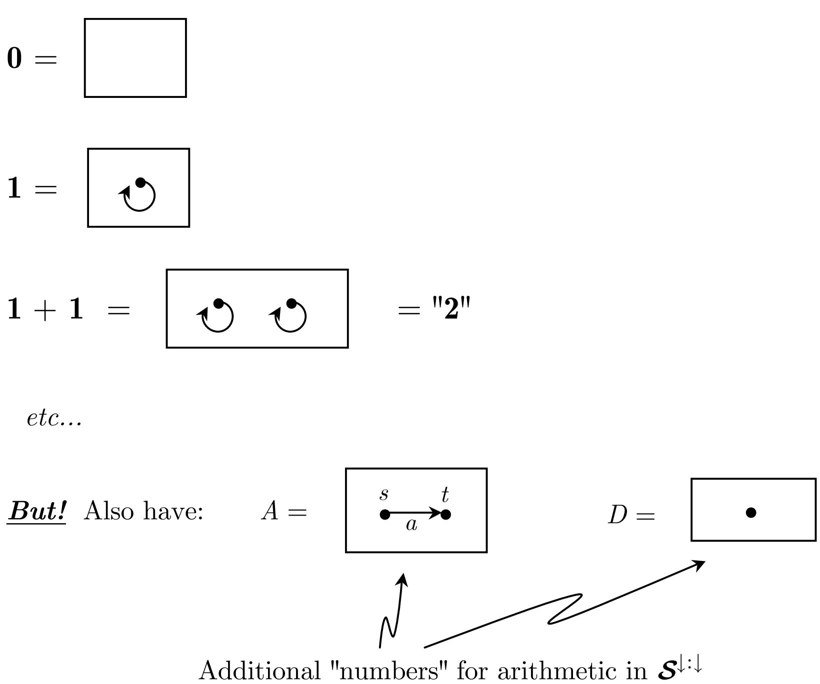
Whatever f_1, f_2 do, f has to replicate.

2. Sums Involving Terminal Objects

Set case $\boldsymbol{\mathcal{S}}$



Graph case $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$



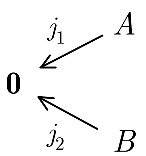
2. "Negative" Objects

Question: What is the *negative* of the number 3?

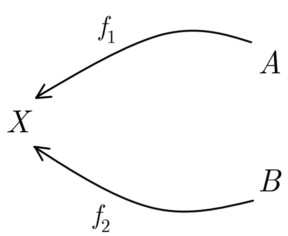
Answer: The solution to $3 + x = 0$, or $x = -3$.

> ***Definition***: Let A be any object in any category with initial object $\mathbf{0}$. Then the ***negative*** of A is the object B such that $A + B = \mathbf{0}$.

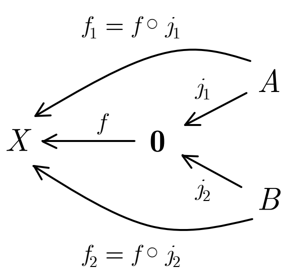
Recall: A sum $\mathbf{0} = A + B$ is an object $\mathbf{0}$ together with a pair of maps $j_1 : A \to \mathbf{0}$, $j_2 : A \to \mathbf{0}$



such that for any object X with maps $f_1 : A \to X$, $f_2 : B \to X$



there is exactly one $X \xleftarrow{f} \mathbf{0}$ such that $f_1 = f \circ j_1$ and $f_2 = f \circ j_2$.



Note: Only one $f : \mathbf{0} \to X$ ($\mathbf{0}$ is the initial object)

So: Only *one* pair (f_1, f_2) (since j_1, j_2 are injective)

So: Only *one* $f_1 : A \to X$ and only *one* $f_2 : B \to X$.

So: A, B must both be the initial object, too!

Thus; If $A + B = \mathbf{0}$ in any category, then $A = B = \mathbf{0}$! Only initial objects have negatives.

Similarly: Can prove that if $A \times B = \mathbf{1}$, then $A = B = \mathbf{1}$.