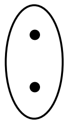
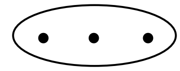
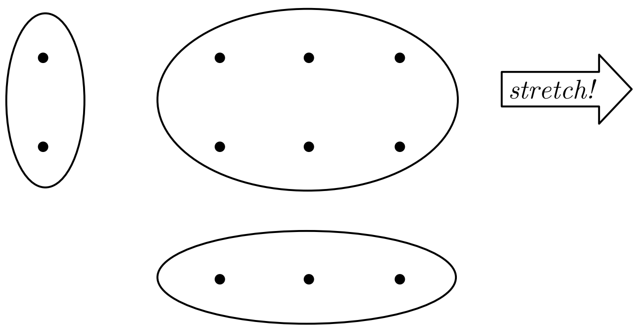
16: Products

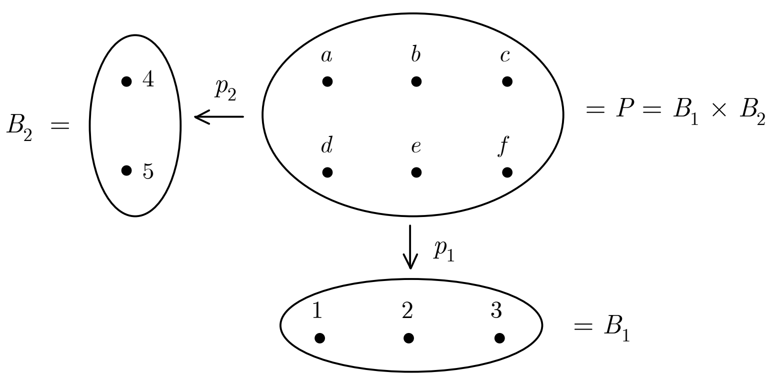
1. Products of Objects

Consider: What does it mean to multiply 2 by 3, when 2 and 3 are represented as sets?

Take the set  and reproduce it 3 times: "stretch" it over the set 



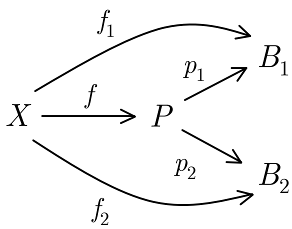
Now consider how elements of the "stretched" set relate to elements of the base sets.
Organize them in the following way:



Any element of P, say a, represents a pair of elements, $a = (4, 1)$, taken from the base sets B_1, B_2.
The "projection" map $p_1 : P \to B_1$ takes a to 1. $p_1(a) = 1$.
The "projection" map $p_2 : P \to B_2$ takes a to 4. $p_2(a) = 4$.

The "product" of B_1 and B_2, then, must be not only the big set P, but *also* these projection maps p_1, p_2.

Definition. A ***product*** of two objects B_1, B_2 is an object P *together* with a pair of maps $p_1 : P \to B_1$ and $p_2 : P \to B_2$, such that, for any object X with maps $f_1 : X \to B_1$, $f_2 : X \to B_2$, there is *exactly one* map $f : X \to P$ for which $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.



Call P, "$B_1 \times B_2$"

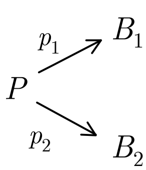
Think of P as consisting of *pairs* of elements (b_1, b_2), one from B_1 and one from B_2. The "projection" maps p_1, p_2 take each element of a given pair to its home.

So: P is supposed to have this "internal" pair-structure to its elements.

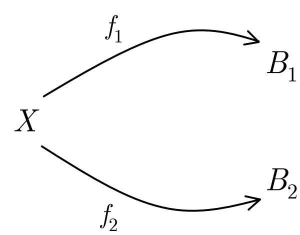
But: We can't directly talk about "internal" elements of an object in category theory! So we need to construct the right external "probe" X that encodes the "internal" pair structure of P.

Require: If some X gets mapped to both B_1 and B_2, there must be only one way to map it to P; namely, the way that "respects" the P-pairs.

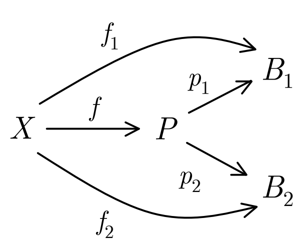
So:



is a product just when, for every

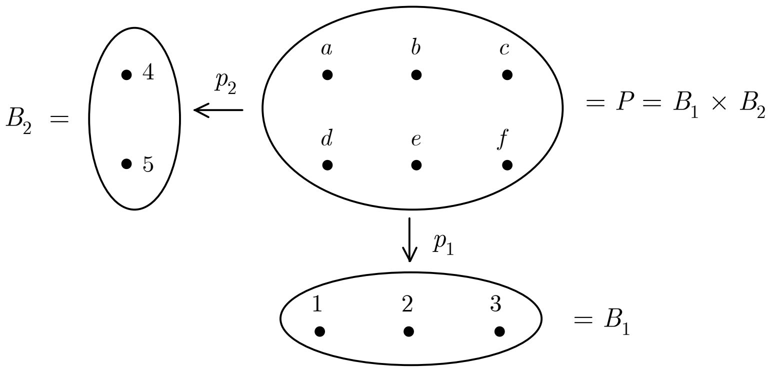


there is exactly one $X \xrightarrow{f} P$ that makes the pieces "fit together" (*i.e.*, "commute"):



Important Note: A product is not *just* an object P, but P together with two maps p_1, p_2 *and* two other objects B_1, B_2. A product is a kinda triangley-shaped figure.

Example 1: A product in $\mathcal{S}$.

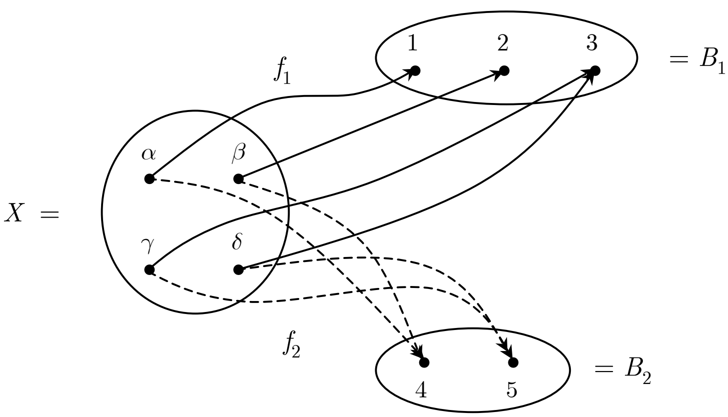


$p_2(a) = p_2(b) = p_2(c) = 4$
$p_2(d) = p_2(e) = p_2(f) = 5$

$p_1(a) = p_1(d) = 1$
$p_1(b) = p_1(e) = 2$
$p_1(c) = p_1(f) = 3$

Is this a product? For *any* other set X with maps $f_1: X \to B_1$, $f_2: X \to B_2$, there is *exactly one* map $f: X \to P$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$?

Check for a simple case:



Does this entail there is exactly one $f: X \to P$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$?

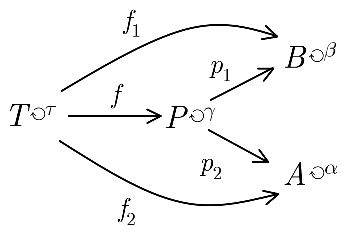
Check:

1. Require $f_1(x) = p_1(f(x))$ for all x in X.
   $f_1(\alpha) = 1 = p_1(f(x)) \Rightarrow f(\alpha) = a$ or d.
   $f_1(\beta) = 2 = p_1(f(x)) \Rightarrow f(\beta) = b$ or e.
   $f_1(\gamma) = 3 = p_1(f(x)) \Rightarrow f(\gamma) = c$ or f.
   $f_1(\delta) = 3 = p_1(f(x)) \Rightarrow f(\delta) = c$ or f.

2. Require $f_2(x) = p_2(f(x))$ for all x in X.
   $f_2(\alpha) = 4 = p_2(f(x)) \Rightarrow f(\alpha) = a$ or b or c.
   $f_2(\beta) = 4 = p_2(f(x)) \Rightarrow f(\beta) = a$ or b or c.
   $f_2(\gamma) = 5 = p_2(f(x)) \Rightarrow f(\gamma) = d$ or e or f.
   $f_2(\delta) = 5 = p_2(f(x)) \Rightarrow f(\delta) = d$ or e or f.

So: $f(\alpha) = a$, $f(\beta) = b$, $f(\gamma) = f$, $f(\delta) = f$ — Only one such f! (The unique f that "respects" the pairs in P.)

How about products in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$?

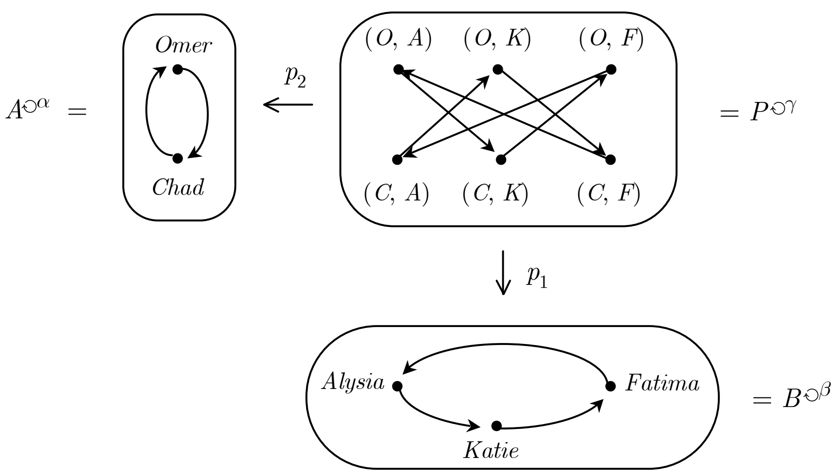


$$\begin{array}{ccc} P & \xrightarrow{p_2} & A \\ \gamma\downarrow & & \downarrow\alpha \\ P & \xrightarrow[p_2]{} & A \end{array} \qquad p_2 \circ \gamma = \alpha \circ p_2$$

$$\begin{array}{ccc} P & \xrightarrow{p_1} & B \\ \gamma\downarrow & & \downarrow\beta \\ P & \xrightarrow[p_1]{} & B \end{array} \qquad p_1 \circ \gamma = \beta \circ p_1$$

$P^{\circlearrowleft\gamma}$ consists of pairs $(a,\, b)$, a in A and b in B, such that $\gamma(a,\, b) = (\alpha(a),\, \beta(b))$!

<u>Example 2.</u>



$\gamma(O,\, A) = (\alpha(O),\, \beta(A)) = (C,\, K)$
$\gamma(C,\, K) = (\alpha(C),\, \beta(K)) = (O,\, F)$
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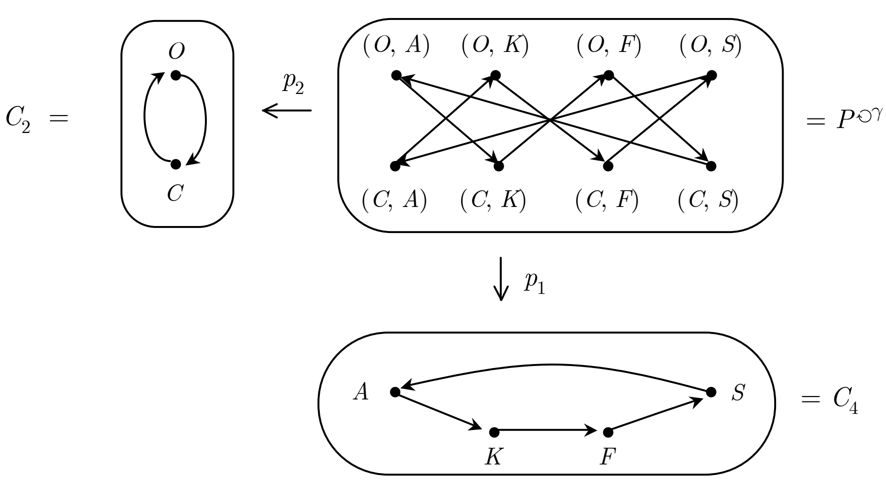
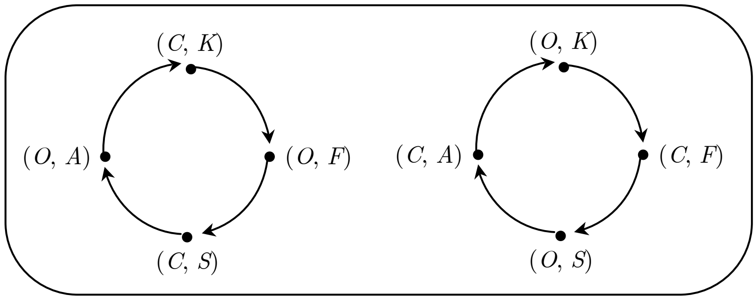
<u>So</u>: $\quad C_2 \times C_3 = C_6$!

<u>Example 3.</u> How about $C_2 \times C_4$?



Note that $P^{\circlearrowleft\gamma}$ can be re-arranged into:

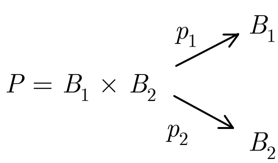
$P^{\circlearrowleft\gamma} =$ 

$= 2C_4$

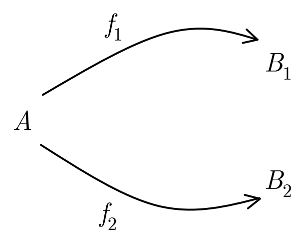
So: $C_2 \times C_4 = 2C_4$

2. Calculating Products

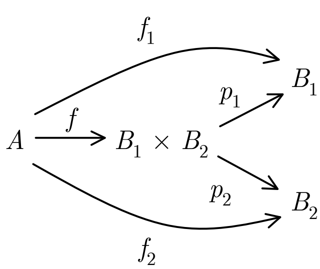
Again:



is a product just when, for every



there is exactly one $A \xrightarrow{f} B_1 \times B_2$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.



$$f(a) = (f_1(a), f_2(a))$$

Another way to say this:

$$\frac{A \to B_1 \times B_2}{A \to B_1,\ A \to B_2}$$

The maps $A \to B_1 \times B_2$...

... correspond to the pairs of maps $A \to B_1$, $A \to B_2$.

Upshot: We can determine the product $B_1 \times B_2$ as soon as we've determined the maps $A \to B_1 \times B_2$, and thus as soon as we've determined the *pairs* of maps $A \to B_1$, $A \to B_2$.

Now: Suppose we let A be the *separating object.*

1. Set case: $\boldsymbol{\mathcal{S}}$

In $\boldsymbol{\mathcal{S}}$ the separating object is the terminal object, $\mathbf{1}$.

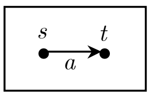
$$\frac{\mathbf{1} \to B_1 \times B_2}{\mathbf{1} \to B_1,\ \mathbf{1} \to B_2}$$

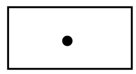
The points of a set product...

... correspond to pairs of points of its "factors"

2. Graph case: $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$

In $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$ the separating objects are the "generic arrow" graph A and the "generic dot" graph D:

$A =$ 

$D =$ 

Claim: To calculate any product of graphs $B_1 \times B_2$, just need to calculate $A \to B_1 \times B_2$ and $D \to B_1 \times B_2$.

Example: Calculate $A \times A = A^2$.

First: Find the dots of A^2

$$\frac{D \to A^2}{D \to A,\ D \to A}$$

The dots of A^2...

... are pairs of dots of A

So: A^2 has 4 dots: $(s,\ t),\ (s,\ s),\ (t,\ s),\ (t,\ t)$

Second: Find the arrows of A^2

$$\frac{A \to A^2}{A \to A,\ A \to A}$$

The arrows of A^2...

... are pairs of arrows of A

So: A^2 has 1 arrow: $(a,\ a)$

Now: Is it a "regular" arrow or a "loop"? Are its source and target dots distinct or the same?

Recall: A loop arrow is a graph point. *So*: How many *points* are there in A^2?

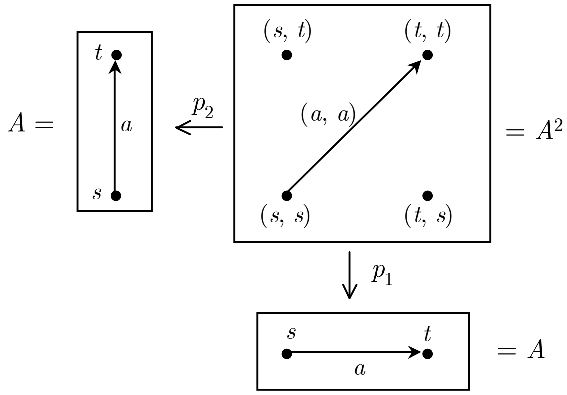
$$\frac{\mathbf{1} \to A^2}{\mathbf{1} \to A,\ \mathbf{1} \to A}$$

The points (loops) of A^2...

... are pairs of points (loops) of A

But there *are* no loops in A. So there can be none in A^2. So the arrow $(a,\ a)$ in A^2 is not a loop!

So:



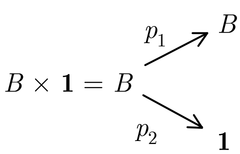
$A^2 = A$ "+" $2D$

define "+" later!

3. Terminal Object as Multiplicative Identity

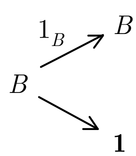
Claim: $B \times \mathbf{1} = B$, for any object B and terminal object $\mathbf{1}$.

Proof: First need to determine the appropriate projection maps:



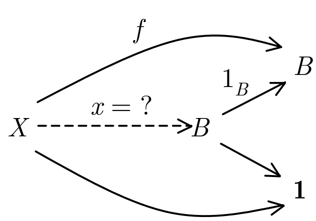
$p_1 = 1_B$
$p_2 =$ unique (since $\mathbf{1}$ is terminal object)

So: Need to demonstrate that



is a product.

Need to show that for any object X, and maps $f: X \to$ B, $X \to \mathbf{1}$, there is just one map $x: X \to B$ such that $1_B \circ x = f$.



Let $x = f$