15: Terminal Objects and Initial Objects

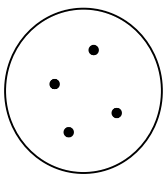
Topics
1. Terminal Objects
2. Points
3. Map-Separating Objects
4. Initial Objects

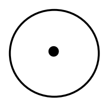
1. Terminal Objects

> ***Definition:*** An object S of a category $\boldsymbol{\mathcal{C}}$ is a terminal object of $\boldsymbol{\mathcal{C}}$ if for each object X of $\boldsymbol{\mathcal{C}}$, there is exactly one $\boldsymbol{\mathcal{C}}$-map $X \to S$.

Question 1: What is a terminal object in $\boldsymbol{\mathcal{S}}$ (category of finite sets)?
Must be a set, such that any other set has only one map to it!

Consider:

$X =$ 

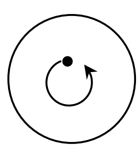
$\mathbf{1} =$ 

How many maps $X \to \mathbf{1}$? (exactly one!)

So: $\mathbf{1}$ is a terminal object in $\boldsymbol{\mathcal{S}}$.

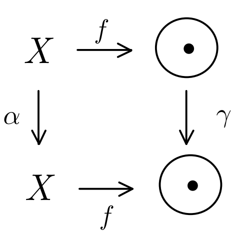
Question 2: What is a terminal object in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ (category of endomaps of sets)?

Consider:

$T^{\circlearrowleft\gamma} =$ 

Check: For any other $X^{\circlearrowleft\alpha}$ in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$, is there only one $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map $X^{\circlearrowleft\alpha} \to T^{\circlearrowleft\gamma}$?

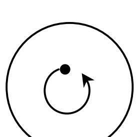
We know: Only one $\boldsymbol{\mathcal{S}}$-map $X \xrightarrow{f} \mathbf{1}$ for any X.



Is this unique $\boldsymbol{\mathcal{S}}$-map f also an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map? If so, it must satisfy

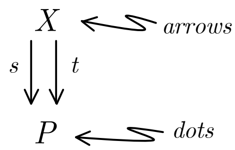
$$f \circ \alpha = \gamma \circ f$$

Claim: It does.
Because: $f \circ \alpha$ is a map $X \to \mathbf{1}$, and $\gamma \circ f$ is a map $X \to \mathbf{1}$, and there is only one such map (since $\mathbf{1}$ is a terminal object of $\boldsymbol{\mathcal{S}}$).

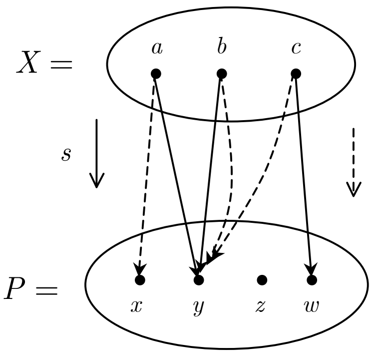
So: $T^{\circlearrowleft\gamma} =$  is a terminal object in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$.

Question 3: What is a terminal object in $\mathcal{S}^{\downarrow:\downarrow}$ (category of irreflexive graphs)?

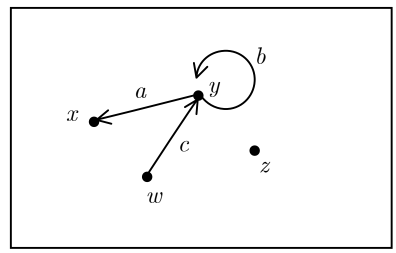
Recall: $\mathcal{S}^{\downarrow:\downarrow}$-object



example:

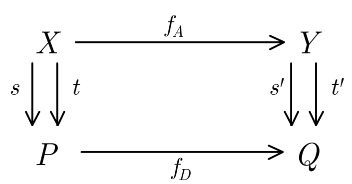


internal diagram



graph

Recall: $\mathcal{S}^{\downarrow:\downarrow}$-map: 2 $\mathcal{S}$-maps (f_A, f_D)

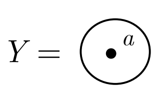


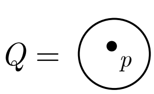
(1) $f_D \circ s = s' \circ f_A$

(2) $f_D \circ t = t' \circ f_A$

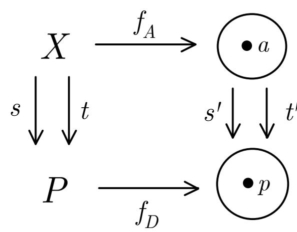
So: The question is: What should the Y/Q object be so that there's only one pair (f_A, f_D)?

How about:





So:



Now Check: Is it the case that the following hold?

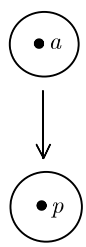
(1) $f_D \circ s = s' \circ f_A$

(2) $f_D \circ t = t' \circ f_A$

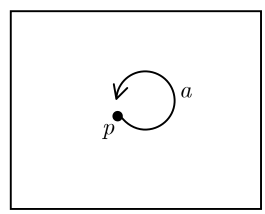
Yes! Since (•p) is a terminal object in S, there is only one **𝓢**-map from X to it. So both (1) and (2) must be true: any two maps from (•p) to X must be the same.

Furthermore: By the same reasoning, $s' = t'$.

So: A terminal object for $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$ is



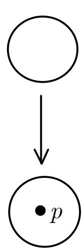
or



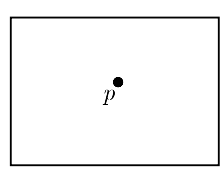
internal diagram *graph*

Any other terminal objects in $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$?

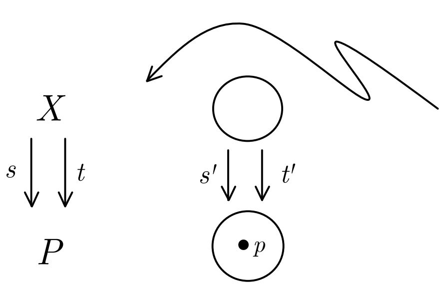
How about the single-dot graph (with no arrows)?



or



Check: Is there exactly one map from any $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-object to the single-dot graph?



Problem! *No* **𝓢**-maps from any set X to the empty set!

So: There can be no $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps from any $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-object to the single-dot graph.

So: The single-dot graph is not a terminal object of $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$.

Theorem: (In any category, the terminal object is "unique up to isomorphism".)

Suppose $\mathcal{C}$ is any category and T_1, T_2 are both terminal objects in $\mathcal{C}$. Then T_1 and T_2 are isomorphic: There are maps $T_1 \xrightarrow{f} T_2$, $T_2 \xrightarrow{g} T_1$ such that $g \circ f = 1_{T1}$, and $f \circ g = 1_{T2}$.

Proof: We're given that T_1, T_2 are terminal objects.

So:

$T_1 \xrightarrow{f} T_2$ is *unique* (since T_2 is terminal).

$T_2 \xrightarrow{g} T_1$ is *unique* (since T_1 is terminal).

So:

$T_1 \xrightarrow{g \circ f} T_1$ is *unique* (since both f and g are unique).

$T_2 \xrightarrow{f \circ g} T_2$ is *unique* ((since both f and g are unique).

So: Since the identities on T_1 and T_2 must exist, it must be that $g \circ f = 1_{T1}$, and $f \circ g = 1_{T2}$.

2. Points of an Object

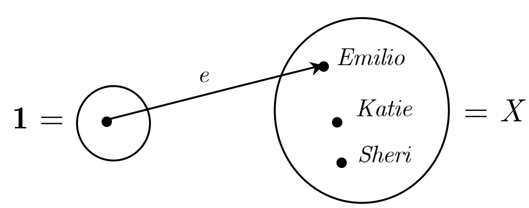
Definition: A point of an object X in any category $\mathcal{C}$ is a map $T \to X$ where T is the terminal object in $\mathcal{C}$.

Recall: In $\mathcal{S}$, points of a set X are maps

$$\mathbf{1} \to X$$

terminal object in $\mathcal{S}$

example:



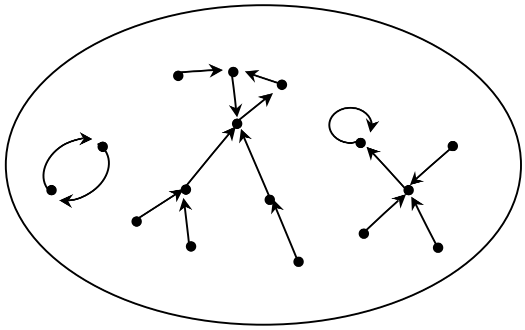
$\mathbf{1} \xrightarrow{e} X$ is a point of X.

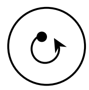
Points of a set are just its elements.

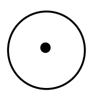
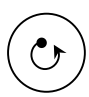
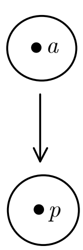
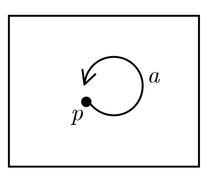
What about points in other categories?

What are the points of an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object?

example:

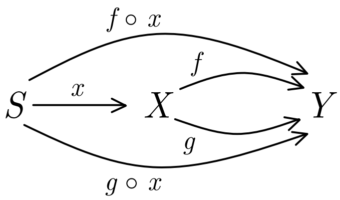
$X^{\circlearrowleft\alpha} =$ 

Points of $X^{\circlearrowleft\alpha}$ are maps from $T^{\circlearrowleft\gamma} =$  to $X^{\circlearrowleft\alpha}$. (Just one such map!)

Category	***Terminal Object***	***"Point of X" means***
$\boldsymbol{\mathcal{C}}$	T	map $T \to X$
$\boldsymbol{\mathcal{S}}$		element of X
$\boldsymbol{\mathcal{S}}^{\circlearrowleft}$		fixed point of X
$\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$	 *or* 	loop arrow of X

3. Map-Separating Objects

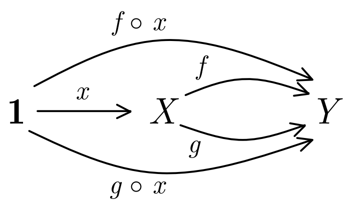
Definition: An object S in a category $\boldsymbol{\mathcal{C}}$ *separates* $\boldsymbol{\mathcal{C}}$-maps just when, for any $\boldsymbol{\mathcal{C}}$-objects X, Y and any $\boldsymbol{\mathcal{C}}$-maps $f: X \to Y$, $g: X \to Y$, and $x: S \to X$, if $f \circ x = g \circ x$, then $f = g$.



If f and g agree on all "generalized elements" x of X, then $f = g$.

Example: In $\boldsymbol{\mathcal{S}}$, the terminal object $\mathbf{1}$ separates $\boldsymbol{\mathcal{S}}$-maps.

Suppose $f: X \to Y$ and $g: X \to Y$ are two maps. If they agree on all *points* of X, then they are identical. In other words, if $f \circ x = g \circ x$ for all *points* $x: \mathbf{1} \to X$, then $f = g$.



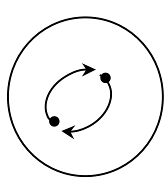
In $\boldsymbol{\mathcal{S}}$, if f and g agree on all points (*i.e.*, *elements*) x of X, then $f = g$.

BUT: The terminal objects in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ and $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$ do *not* separate maps!

Why?: Objects in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ and $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$ may not have points!

example 1. (Any $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object without fixed points)

$X^{\circlearrowleft\alpha} =$

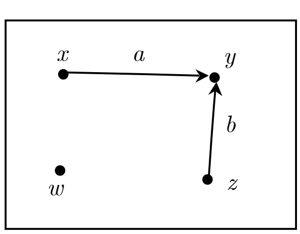


No points! (No "fixed points".)

So: Two $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps $f: X^{\circlearrowleft\alpha} \to Y^{\circlearrowleft\beta}$, $g: X^{\circlearrowleft\alpha} \to Y^{\circlearrowleft\beta}$ with $f \neq g$ will trivially agree on all points of $X^{\circlearrowleft\alpha}$ (since there are none).

example 2. (Any $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-object without loop arrows)

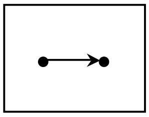
$X =$

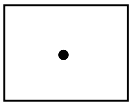


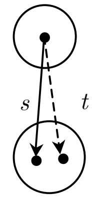
No points! (No "loop arrows".)

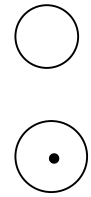
So: Two $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps f, g will trivially agree on all points of X (since there are none).

Map-Separating Objects for $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$

$A =$  *"generic arrow" graph*

$D =$  *"generic dot" graph*

$A =$ 

$D =$ 

internal diagrams of A and D

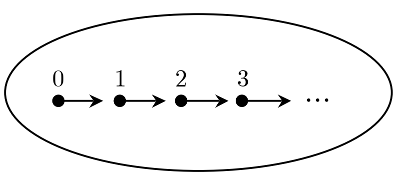
Claim: For any graph X, each arrow in X is given by exactly one $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-map $A \to X$, and each dot in X is given by exactly one $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-map $D \to X$.

So: Suppose $f\colon\ X \to Y$ and $g\colon\ X \to Y$ are any two $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps. If $f \circ x = g \circ x$ for all maps $x\colon\ A \to X$ and all maps $x\colon\ D \to X$, then $f = g$.

In other words, if f and g agree on all arrows and dots in X, then $f = g$!

Map-Separating Object for $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$

A bit trickier to visualize. For two $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps $f, g\colon\ X^{\circlearrowleft\alpha} \to Y^{\circlearrowleft\beta}$ to be identical, they have to agree on all "generalized elements" of $X^{\circlearrowleft\alpha}$. How do we identify these generalized elements? Consider the $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $\mathbb{N}^{\circlearrowleft\sigma}$:

$\mathbb{N}^{\circlearrowleft\sigma} =$ 

$\mathbb{N} =$ *set of natural numbers* $\{0, 1, 2, 3, ...\}$

σ *is the "successor" map* $\sigma(n) = n + 1$

Recall: $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps from $\mathbb{N}^{\circlearrowleft\sigma}$ to any $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $X^{\circlearrowleft\alpha}$ name all the elements of $X^{\circlearrowleft\alpha}$. These $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps are the "generalized elements" for $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-objects!

So: $\mathbb{N}^{\circlearrowleft\sigma}$ is a separating object for $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$:

Suppose $f, g\colon\ X^{\circlearrowleft\alpha} \to Y^{\circlearrowleft\beta}$ are any two $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps. If $f \circ x = g \circ x$ for all maps $x\colon\ \mathbb{N}^{\circlearrowleft\sigma} \to X^{\circlearrowleft\alpha}$, then $f = g$.

4. Initial Objects

Definition: S is an *initial object* of a category $\boldsymbol{\mathcal{C}}$ if for every $\boldsymbol{\mathcal{C}}$-object X there is exactly one $\boldsymbol{\mathcal{C}}$-map $S \to X$.

"dual" of terminal object (the "reverse" of the definition of terminal object: exchange domain and codomain)

Claim: In $\boldsymbol{\mathcal{S}}$, the empty set is the initial object.

$\mathbf{0} =$  $= X$

Proof: There is exactly *one* map $\mathbf{0} \to X$. Since there are no elements in $\mathbf{0}$, it's trivially true that every element of $\mathbf{0}$ gets assigned exactly one element of X (and there's only one way to do this; namely the way in which there are no arrows between $\mathbf{0}$ and X).

Claim: In $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ and $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$, the initial objects are also the empty set.

Proof: *First Check*: $\mathbf{0}$ is an object in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ (why?). And $\mathbf{0}$ is also an object in $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$ (why?).

Then: Since there is only one $\boldsymbol{\mathcal{S}}$-map from $\mathbf{0}$ to any set, and $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps and $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps are particular types of $\boldsymbol{\mathcal{S}}$-maps, there will only be one $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map between $\mathbf{0}$ and any $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object, and there will only be one $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-map between $\mathbf{0}$ and any $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-object.