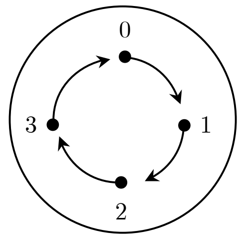
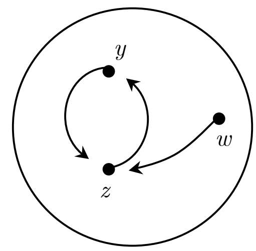
# 14: Generalized Elements in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$

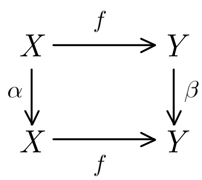
## ***Structure-preserving maps from a cycle to another endomap***

Let $X^{\circlearrowleft\alpha}$ and $Y^{\circlearrowleft\beta}$ be the $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-objects (*i.e.*, dynamical systems):

$X^{\circlearrowleft\alpha} =$ 


$Y^{\circlearrowleft\beta} =$ 


Find an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map $X^{\circlearrowleft\alpha} \xrightarrow{\;f\;} Y^{\circlearrowleft\beta}$ such that $f(0) = y$.




$f \circ \alpha = \beta \circ f$

*or*

$f(\alpha(x)) = \beta(f(x))$, for all $x$ in $X$

To specify $f$, we need to say what it does to each element $x$ of $X^{\circlearrowleft\alpha}$:

$x = 0$: $\quad f(\alpha(0)) = \beta(f(0))$, or $f(1) = \beta(y) = z$

$x = 1$: $\quad f(\alpha(1)) = \beta(f(1))$, or $f(2) = \beta(z) = y$

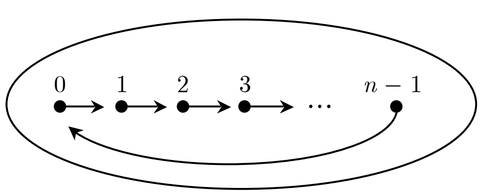
$x = 2$: $\quad f(\alpha(2)) = \beta(f(2))$, or $f(3) = \beta(y) = z$

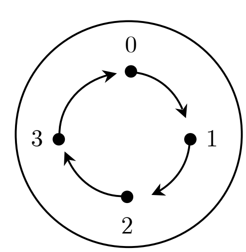
$x = 3$: $\quad f(\alpha(3)) = \beta(f(3))$, or $f(0) = \beta(z) = y$

*How many other maps are there?*
*(Only other one takes 0 to z.)*

> ***Definition***: An element $x$ of an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $X^{\circlearrowleft\alpha}$ has period $n$ just when $\alpha^n(x) = x$.

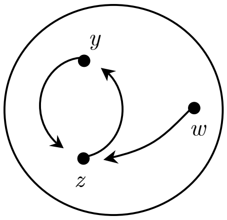
> ***Definition***: For any natural number $n$, the cycle of length $n$, $C_n$, is the set of $n$ elements $\{0, 1, 2, ..., n\}$ with the "successor" endomap, with the successor of $n - 1$ being 0.

$C_n =$ 


$C_4 =$ 

***Note***: $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps $C_4 \xrightarrow{f} Y^{\circlearrowleft\beta}$ correspond to all elements of $Y^{\circlearrowleft\beta}$ with period 4!

$Y^{\circlearrowleft\beta} =$




Two elements with period 4

$y$: $\beta^4(y) = y$

$z$: $\beta^4(z) = z$

Two maps $C_4 \xrightarrow{f} Y^{\circlearrowleft\beta}$:

$f_1(0) = y,\ f_1(1) = z,\ f_1(2) = y,\ f_1(3) = z$

$f_2(0) = z,\ f_2(1) = y,\ f_2(2) = z,\ f_2(3) = y$

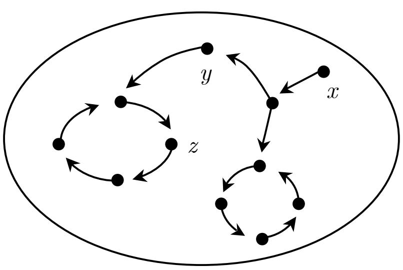
***In general***: For any arbitrary $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $Y^{\circlearrowleft\beta}$, the $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps $C_n \xrightarrow{f} Y^{\circlearrowleft\beta}$ correspond to all elements of $Y^{\circlearrowleft\beta}$ with period $n$.

***Terminology***: The $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps $C_n \xrightarrow{f} Y^{\circlearrowleft\beta}$ ***name*** the elements of $Y^{\circlearrowleft\beta}$ with period $n$.

***Question***: How can we name ***arbitrary*** or ***"generalized"*** elements of an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object?

*example 1*:

$Y^{\circlearrowleft\beta} =$

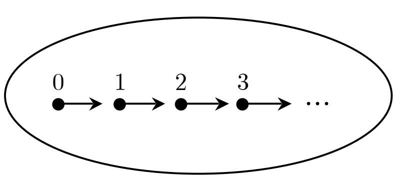



$\beta^4(z) = z$; *so* $z$ has period 4

$x$ has no period; but $x$ has the "positive property" of "*being two steps away from a 4-cycle*"

*example 2:*

$\mathbb{N}^{\circlearrowleft\sigma} =$




$\mathbb{N}$ = set of natural numbers {0, 1, 2, 3, ...}

$\mathbb{N} \xrightarrow{\sigma} \mathbb{N}$ is the "successor" map $\sigma(n) = n + 1$

0 has no positive properties

**Claim**: $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps from $\mathbb{N}^{\circlearrowleft\sigma}$ to any $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $Y^{\circlearrowleft\beta}$ name *all* the elements of $Y^{\circlearrowleft\beta}$.

***In particular***: For each element $y$ of $Y^{\circlearrowleft\beta}$, there is a unique map $\mathbb{N}^{\circlearrowleft\sigma} \xrightarrow{f} Y^{\circlearrowleft\beta}$ such that $f(0) = y$.

***Proof***. Let $\mathbb{N}^{\circlearrowleft\sigma} \xrightarrow{f} Y^{\circlearrowleft\beta}$ be an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map such that $f(0) = y$ for element $y$ of $Y^{\circlearrowleft\beta}$.

Now: Show that any other $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map $\mathbb{N}^{\circlearrowleft\sigma} \xrightarrow{g} Y^{\circlearrowleft\beta}$ is such that, if $g(0) = f(0)$, then $g = f$.

Given: (1) $f \circ \sigma = \beta \circ f$
(2) $g \circ \sigma = \beta \circ g$
(3) $g(0) = f(0)$

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & Y \\ \sigma\downarrow & & \downarrow\beta \\ \mathbb{N} & \xrightarrow[f]{} & Y \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \xrightarrow{g} & Y \\ \sigma\downarrow & & \downarrow\beta \\ \mathbb{N} & \xrightarrow[g]{} & Y \end{array}$$

Then:
$f(1) = f(\sigma(0))$ *given (definition of $\sigma$)*
$= \beta(f(0))$ *given (1)*
$= \beta(g(0))$ *given (3)*
$= g(\sigma(0))$ *given (2)*
$= g(1)$ *given (definition of $\sigma$)*

So: If $f$ and $g$ agree on 0, then they agree on 1.
Now: Suppose for any $n$, $g(n) = f(n)$. Call this assumption (3′).
Does this then entail $f(n + 1) = g(n + 1)$?

Check:
$f(n + 1) = f(\sigma(n))$ *given (definition of $\sigma$)*
$= \beta(f(n))$ *given (1)*
$= \beta(g(n))$ *given (3′)*
$= g(\sigma(n))$ *given (2)*
$= g(n + 1)$ *given (definition of $\sigma$)*

So: We've shown that if $f(0) = g(0)$, then $f(1) = g(1)$. And if $f(n) = g(n)$ for any $n$, then $f(n + 1) = g(n + 1)$. This means, if $f$ and $g$ agree on 0, then they agree on 1, and hence 2, and hence 3, *etc*. So they agree on *all* elements of $\mathbb{N}$.

So: $f = g$!