14: More Categories

Topics
1. The Category of Endomaps of Sets
2. The Category of Graphs

Recall: So far we've been talking about the category of sets, call it $\boldsymbol{\mathcal{S}}$.

I. $\boldsymbol{\mathcal{S}}$-*objects.* Sets

II. $\boldsymbol{\mathcal{S}}$-*maps.* Maps between sets. Each is defined by specifying:

(1) a *domain* set A

(2) a *codomain* set B

(3) a *rule*: Each point in A gets assigned exactly one point in B.

III. *Identity maps.* One for each $\boldsymbol{\mathcal{S}}$-object.

IV. *Composite maps.* One for each pair of $\boldsymbol{\mathcal{S}}$-maps f, g for which the codomain of one is the domain of the other.

V. *Identity laws.*

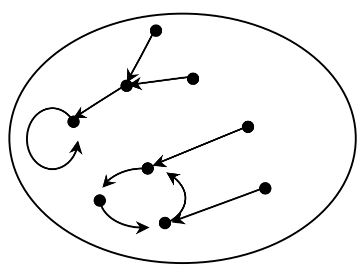
VI. *Associative law.*

1. The Category $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ of Endomaps of Sets

I. $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-*objects.* Any set equipped with an endomap. ← *A set "structured" by an endomap.*

$$X \xrightarrow{\alpha} X \qquad \textit{or} \qquad X^{\circlearrowleft\alpha}$$

example:

$X^{\circlearrowleft\alpha} =$ 

Check: *Does this define a map? Yes: Every point gets assigned exactly one other point.*

II. $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-*maps.* These must go from one $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $X^{\circlearrowleft\alpha}$ to another $Y^{\circlearrowleft\beta}$ such that the "structure" of the endomaps is preserved. So:

> ***Defintion***. An $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ is an $\mathcal{S}$-map $X \xrightarrow{f} Y$ that satisfies $f \circ \alpha = \beta \circ f$.

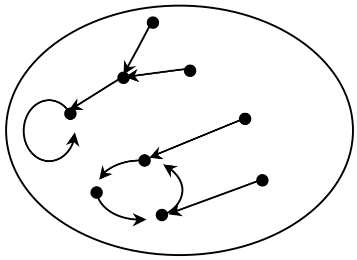
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha\downarrow & & \downarrow\beta \\ X & \xrightarrow[f]{} & Y \end{array}$$

$f \circ \alpha = \beta \circ f$.

Again, this means: "If you first do α and then f, it should be the same as first doing f and then β." This constraint entails that f preserves the ***structure*** of α in the structure of β; *i.e.*, whatever α does in X, β will "mirror" in Y.

The structure of an endomap α can be represented by the internal diagram of $X^{\circlearrowleft\alpha}$.

example:

$X^{\circlearrowleft\alpha} =$ 

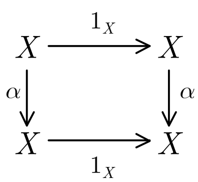
- # fixed points = 1
- # pts 1 step away from a fixed pt = 1
- # pts 2 steps away from a fixed pt = 2
- # 3-cycles = 1
- # pts 1 step away from a 3-cycle = 2
- *etc...*

ASIDE: A ***fixed point*** x of a map α is a point in the domain of α for which $\alpha(x) = x$; *i.e.*, applying α to a fixed point yields that same point again.

This is the kind of structure preserved by $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps. If an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $X^{\circlearrowleft\alpha}$ has it, and $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ is an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map, then the $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $Y^{\circlearrowleft\beta}$ has it, too.

III. *Identity maps* in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$.

For $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-object $X^{\circlearrowleft\alpha}$, take as identity map $X^{\circlearrowleft\alpha} \xrightarrow{1_X} X^{\circlearrowleft\alpha}$.



Check: Must be an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map; *i.e.*, must have the property $1_X \circ \alpha = \alpha \circ 1_X$

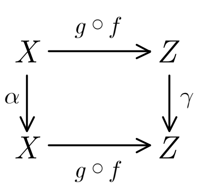
$1_X \circ \alpha = \alpha$ *identity law for* $\boldsymbol{\mathcal{S}}$

$\overset{\checkmark}{=} \alpha \circ 1_X$ *identity law for* $\boldsymbol{\mathcal{S}}$

Note: We can use the $\boldsymbol{\mathcal{S}}$-identity laws here because we're dealing with $\boldsymbol{\mathcal{S}}$-maps ($\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps are certain types of $\boldsymbol{\mathcal{S}}$-maps)

IV. *Composition of maps* in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$.

For $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ and $Y^{\circlearrowleft\beta} \xrightarrow{g} Z^{\circlearrowleft\gamma}$ let the composite of f and g be $X^{\circlearrowleft\alpha} \xrightarrow{g\circ f} Z^{\circlearrowleft\gamma}$.



Check: Must be an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map; *i.e.*, must have the property $(g \circ f) \circ \alpha = \gamma \circ (g \circ f)$

$(g \circ f) \circ \alpha = g \circ (f \circ \alpha)$	*associative law for* $\boldsymbol{\mathcal{S}}$
$= g \circ (\beta \circ f)$	*given (i.e., f is an* $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$*-map)*
$= (g \circ \beta) \circ f$	*associative law for* $\boldsymbol{\mathcal{S}}$
$= (\gamma \circ g) \circ f$	*given (i.e., g is an* $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$*-map)*
$\overset{\checkmark}{=} \gamma \circ (g \circ f)$	*associative law for* $\boldsymbol{\mathcal{S}}$

Now we just have to show that $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps obey the *identity laws* and the *associative law*, and we'll be done constructing $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ as a category. But it should be obvious that $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps do obey these laws, because $\boldsymbol{\mathcal{S}}$-maps obey these laws, and $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-maps are certain types of $\boldsymbol{\mathcal{S}}$-maps.

Question: What is an isomorphism in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$? *Ans.* An $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map that has an inverse. *What does it do in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$?*

In $\boldsymbol{\mathcal{S}}$, isomorphic objects have the *same number of points.*

In $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$, isomorphic objects have:

- *same number of points*
- *same number of fixed points*
- *same number of 2-cycles*
- *same number of points 1 step away from a fixed point*
- *etc...*

(the last four items, bracketed:) *additional structure in $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$*

Three Subcategories of $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$ -- Idempotents, Automorphisms, and Involutions

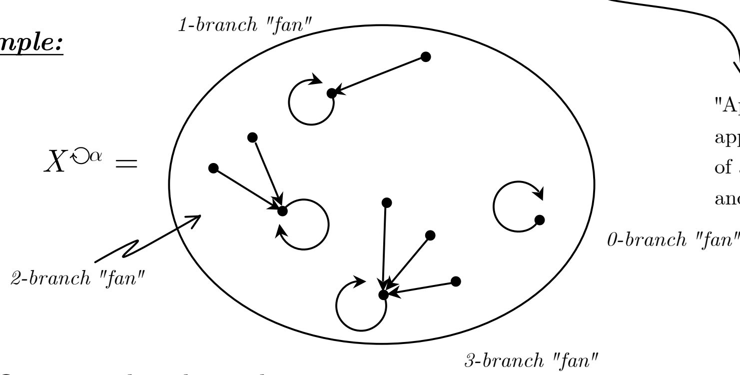
Definition. An $\boldsymbol{\mathcal{S}}$-endomap α is an ***idempotent*** just when $\alpha \circ \alpha = \alpha$.
An $\boldsymbol{\mathcal{S}}$-endomap α is an ***automorphism*** just when α is also an *isomorphism.*
An $\boldsymbol{\mathcal{S}}$-endomap θ for an $\boldsymbol{\mathcal{S}}$-object X is an ***involution*** just when $\theta \circ \theta = 1_X$.

1. Let $\boldsymbol{\mathcal{S}}^e$ be the category of idempotent endomaps on $\boldsymbol{\mathcal{S}}$.

$X^{\circlearrowleft\alpha}$ is an $\boldsymbol{\mathcal{S}}^e$-object just when α is an idempotent; *i.e.*, $\alpha \circ \alpha = \alpha$.

example:

$X^{\circlearrowleft\alpha} =$



"Applying α twice yields the same result as applying α once". This entails the internal diagram of any $\boldsymbol{\mathcal{S}}^e$-object can only contain fixed points and/or points 1 step away from fixed points.

In $\boldsymbol{\mathcal{S}}^e$, isomorphic objects have:

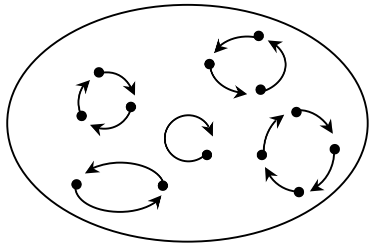
- *same number of points*
- *same number of fixed point "fans"*

2. Let $\boldsymbol{\mathcal{S}}^a$ be the category of automorphisms on $\boldsymbol{\mathcal{S}}$.

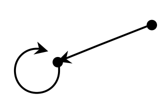
$X^{\circlearrowleft\alpha}$ is an $\boldsymbol{\mathcal{S}}^a$-object just when there is an inverse β of α; *i.e.*, $\alpha \circ \beta = 1_X$ and $\beta \circ \alpha = 1_X$.

example:

$X^{\circlearrowleft\alpha} =$



All α-arrows can be "reversed". This entails the internal diagram of any $\boldsymbol{\mathcal{S}}^a$-object can only have cycles with no branches!
Why? Consider the simplest 1-branch fan:



The result of reversing both arrows cannot represent a map: there would be a point with two arrows coming out of it, which a map cannot do.

In $\boldsymbol{\mathcal{S}}^a$, isomorphic objects have:

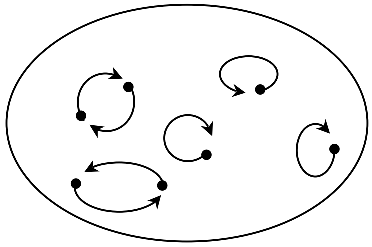
- *same number of points*
- *same number of same-type cycles*

3. Let $\boldsymbol{\mathcal{S}}^{\theta}$ be the category of involutions on $\boldsymbol{\mathcal{S}}$.

$X^{\circlearrowleft\theta}$ is an $\boldsymbol{\mathcal{S}}^{\theta}$-object just when $\theta \circ \theta = 1_X$.

example:

$X^{\circlearrowleft\theta} =$



"Applying θ twice gets you back to where you started." This entails the internal diagram of any $\boldsymbol{\mathcal{S}}^{\theta}$-object can only consist of fixed points and/or 2-cycles!

In $\boldsymbol{\mathcal{S}}^{\theta}$, isomorphic objects have:

- *same number of fixed points*
- *same number of 2-cycles*

Application of $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$: Dynamical Systems

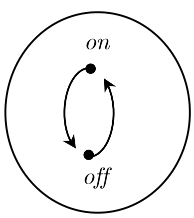
Let $X =$ set of possible states of a system

$\alpha =$ "evolution" map (evolves states in time)

example: Lamp with on/off switch

$X = \{on,\ off\}$

$X^{\circlearrowleft\alpha} =$



$\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ sends a state x of $X^{\circlearrowleft\alpha}$-machine to a state $f(x)$ of $Y^{\circlearrowleft\beta}$-machine that evolves under β in the same way that x evolves under α.

***Exercise #1*, pg. 161**

Suppose $x' = \alpha^3(x)$ and $X^{\circlearrowleft\alpha} \xrightarrow{f} Y^{\circlearrowleft\beta}$ is an $\boldsymbol{\mathcal{S}}^{\circlearrowleft}$-map. Let $y = f(x)$ and $y' = \beta^3(y)$. Then $f(x') = y'$.

<u>Given</u>:

(1) $x' = \alpha(\alpha(\alpha(x)))$ (3) $y' = \beta(\beta(\beta(y)))$

(2) $y = f(x)$ (4) $f \circ \alpha = \beta \circ f$ or $f(\alpha(x)) = \beta(f(x))$, for any x in X

<u>So</u>:

$$
\begin{aligned}
f(x') &= f(\alpha(\alpha(\alpha(x))) && \textit{given } (1)\\
&= \beta(f(\alpha(\alpha(x))) && \textit{given } (4)\\
&= \beta(\beta(f(\alpha(x))) && \textit{given } (4)\\
&= \beta(\beta(\beta(f(x))) && \textit{given } (4)\\
&= \beta(\beta(\beta(y))) && \textit{given } (2)\\
&\overset{\checkmark}{=} y' && \textit{given } (3)
\end{aligned}
$$

2. The Category $\mathcal{S}^{\downarrow:\downarrow}$ of Irreflexive Graphs

I. $\mathcal{S}^{\downarrow:\downarrow}$*-objects.* A pair of $\mathcal{S}$-maps s, t with the same domain and the same codomain.

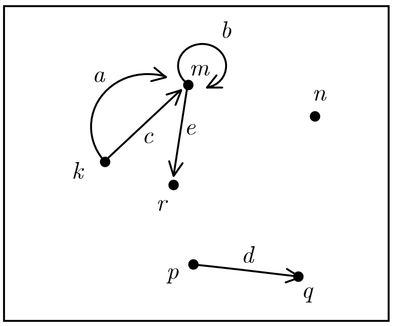
$$X \overset{s}{\underset{t}{\rightrightarrows}} P$$

$s =$ "source" map
$t =$ "target" map

Terminology:
$X =$ set of "arrows"
$P =$ set of "dots"
$s(x) =$ source dot of arrow x in X
$t(x) =$ target dot of arrow x in X

Represents a graph consisting of "arrows" and "dots":

example:



$s(a) = k$	$t(a) = m$
$s(b) = m$	$t(b) = m$
$s(c) = k$	$t(c) = m$
$s(d) = p$	$t(d) = q$
$s(e) = m$	$t(e) = r$

Note: This is *not* an internal diagram of an $\mathcal{S}^{\downarrow:\downarrow}$-object! It's literally a graph: a drawing that could represent cities (dots) and roads connecting them (arrows), for example; or anything else you want. In particular, the "arrows" do not represent maps, and the dots do not represent points. (Unfortunantly, we're using the same term "arrow" for both these graph "arcs" and maps.)

II. $\mathcal{S}^{\downarrow:\downarrow}$*-maps.*

Definition. An $\mathcal{S}^{\downarrow:\downarrow}$-map $\left(X \overset{s}{\underset{t}{\rightrightarrows}} P\right) \xrightarrow{f} \left(Y \overset{s'}{\underset{t'}{\rightrightarrows}} Q\right)$ is a *pair* of $\mathcal{S}$-maps

$X \xrightarrow{f_A} Y$, $P \xrightarrow{f_D} Q$ for which

(1) $f_D \circ s = s' \circ f_A$
(2) $f_D \circ t = t' \circ f_A$

$$\begin{array}{ccc} X & \xrightarrow{f_A} & Y \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle s'}\downarrow\downarrow{\scriptstyle t'} \\ P & \xrightarrow[f_D]{} & Q \end{array}$$

f preserves the source and target relations of the graphs

III. *Identity maps* in $\mathcal{S}^{\downarrow:\downarrow}$.

For $\mathcal{S}^{\downarrow:\downarrow}$-object $X \overset{s}{\underset{t}{\rightrightarrows}} P$ take as identity map the pair of $\mathcal{S}$-maps $X \xrightarrow{1_X} X$ and $P \xrightarrow{1_P} P$.

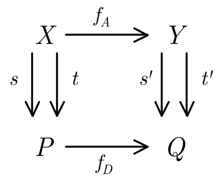
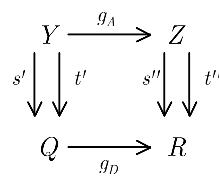
$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} & & {\scriptstyle s}\downarrow\downarrow{\scriptstyle t} \\ P & \xrightarrow[1_P]{} & P \end{array}$$

Check: Must be an $\mathcal{S}^{\downarrow:\downarrow}$-map:

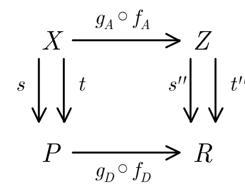
(1) $1_P \circ s = s$ — *identity law for* $\mathcal{S}$
$\overset{\checkmark}{=} s \circ 1_X$ — *identity law for* $\mathcal{S}$

(2) $1_P \circ t = t$ — *identity law for* $\mathcal{S}$
$\overset{\checkmark}{=} t \circ 1_X$ — *identity law for* $\mathcal{S}$

IV. *Composite maps* in $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$.

For $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps  and 

let their composite $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-map be the pair of $\boldsymbol{\mathcal{S}}$-maps $g_A \circ f_A$, $g_D \circ f_D$.



Check: Must be an $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-map; *i.e.*, we must show:

(1) $(g_D \circ f_D) \circ s = s'' \circ (g_A \circ f_A)$

(2) $(g_D \circ f_D) \circ t = t'' \circ (g_A \circ f_A)$

and we're given that (f_A, f_D) and (g_A, g_D) are $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps; *i.e.*, we're given:

(i) $f_D \circ s = s' \circ f_A$ (iii) $g_D \circ s' = s'' \circ g_A$

(ii) $f_D \circ t = t' \circ f_A$ (iv) $g_D \circ t' = t'' \circ g_A$

So:

(1)
$$\begin{aligned}(g_D \circ f_D) \circ s &= g_D \circ (f_D \circ s) && \textit{assoc law for } \boldsymbol{\mathcal{S}}\\ &= g_D \circ (s' \circ f_A) && \textit{given}\\ &= (g_D \circ s') \circ f_A && \textit{assoc law for } \boldsymbol{\mathcal{S}}\\ &= (s'' \circ g_A) \circ f_A && \textit{given}\\ &\overset{\checkmark}{=} s'' \circ (g_A \circ f_A) && \textit{assoc law for } \boldsymbol{\mathcal{S}}\end{aligned}$$

(2)
$$\begin{aligned}(g_D \circ f_D) \circ t &= g_D \circ (f_D \circ t) && \textit{assoc law for } \boldsymbol{\mathcal{S}}\\ &= g_D \circ (t' \circ f_A) && \textit{given}\\ &= (g_D \circ t') \circ f_A && \textit{assoc law for } \boldsymbol{\mathcal{S}}\\ &= (t'' \circ g_A) \circ f_A && \textit{given}\\ &\overset{\checkmark}{=} t'' \circ (g_A \circ f_A) && \textit{assoc law for } \boldsymbol{\mathcal{S}}\end{aligned}$$

Now we just have to show that $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps obey the *identity laws* and the *associative law*, and we'll be done constructing $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$ as a category. But it should be obvious that $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps do obey these laws, because $\boldsymbol{\mathcal{S}}$-maps obey these laws, and $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$-maps are certain types of pairs of $\boldsymbol{\mathcal{S}}$-maps.

Applications of $\boldsymbol{\mathcal{S}}^{\downarrow:\downarrow}$

- *electric circuits*
- *transportation (road systems/towns, etc.)*
- *linguistics (dots = nouns, arrows = verbs, etc.)*
- *conspiracy theories...*