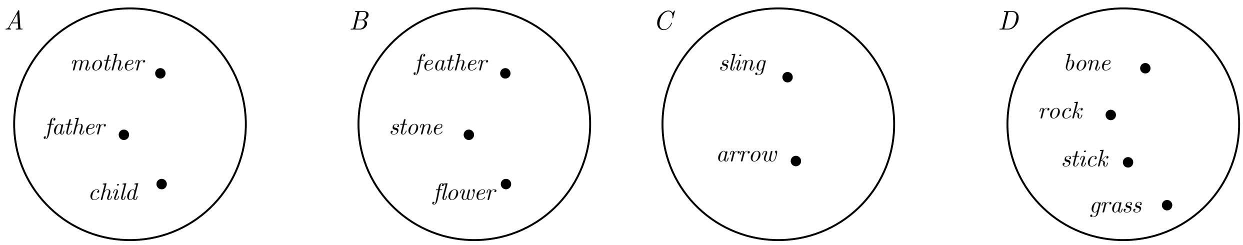
# 12: Article II - Isomorphisms

***Topics***
1. Isomorphisms
2. Retractions and Sections

## 1. Isomorphisms

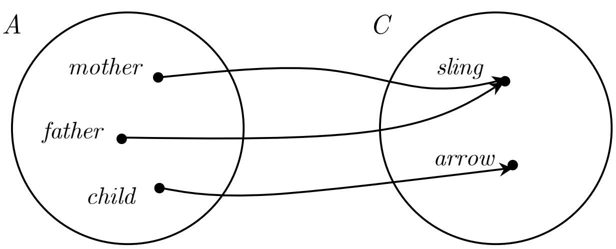
*Idea*: Isomorphisms are maps that preserve "similarity".

Suppose we have 4 collections:

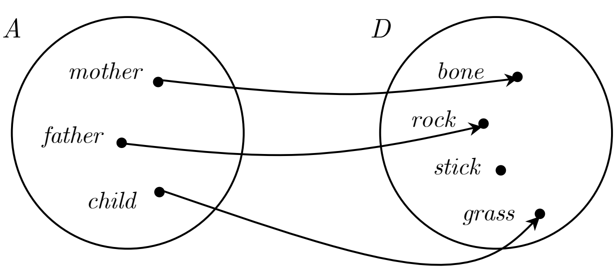



If you didn't know about numbers, how could you say $A$ and $B$ are similar to each other, but not to $C$ or $D$? Do it in terms of ***maps***:

> Can map each element of $A$ to a *unique* element of $B$ such that *no* elements in $B$ are left over.

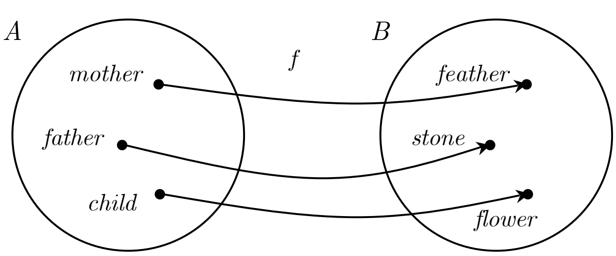



*This map doesn't send elements in A to* ***unique*** *elements in C.*




*This map leaves an element in D left over.*

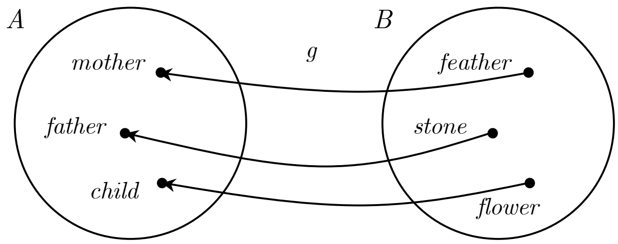
> ***ASIDE***: *Recall the essential characteristic of a map: It must take every element in its domain to exactly one element in its codomain. So each of these 3 examples does define a map.*

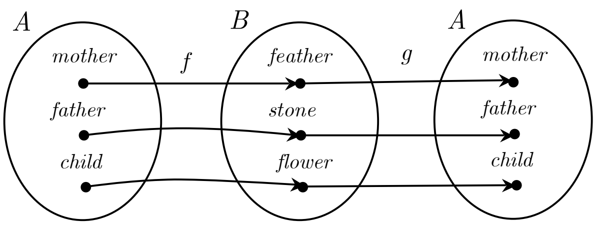



*This map takes each element of A to a unique element of B such that no elements of B are left over.*

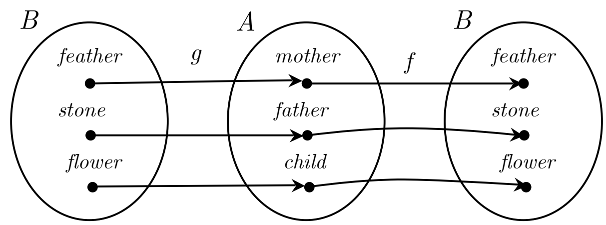
***Call this last map "f".***

Essential characteristic of $f$ - it has an ***inverse map***, call it $g$.




$g \circ f = 1_A$



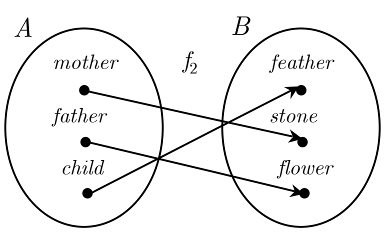

$f \circ g = 1_B$

***Definition***. A map $A \xrightarrow{f} B$ is called an ***isomorphism*** (or *invertible* map) if there is a map $B \xrightarrow{g} A$ for which $g \circ f = 1_A$ and $f \circ g = 1_B$. Such a map $g$ is called an ***inverse*** for $f$.

Two objects $A$ and $B$ are *isomorphic* if there is at least one isomorphism between them.

***Note***: There can be more than one isomorphism between two objects.

***example***:




How many isomorphisms are there in this case?

($3^3 = 27$ possible maps, of which 6 are isomorphisms!*)

What is the inverse for $f_2$?

****ASIDE***: We (should) understand how to calculate the total number of possible maps between any given domain and codomain (homework assignment). For the number of particular maps that are *isomorphisms*, consider the following. Suppose there are $n$ elements in the domain. The game is to hook each one of them up with a unique element in the codomain, which in this case has the same number $n$ of elements, so that no elements in the codomain are left over. *So*:
There are $n$ different ways to hook up the first element.
Once the first is hooked up, there are $n - 1$ different ways to hook up the second.
There are then $n - 2$ different ways to hook up the third, *etc*....
When we get to the last, there will only be one way to hook it up (there will only be one element in the domain left).
So there will be a total of $n \times (n-1) \times (n-2) \times .... \times 1$ different ways to play this game. For instance, when there are $n = 3$ elements in the domain, there will be $3 \times (3-1) \times (3-2) = 6$ different ways to hook up elements. In other words, 6 different isomorphisms. (The expression $n \times (n-1) \times (n-2) \times .... \times 1$ is called "$n$ factorial" and is abbreviated by $n!$.)

***Exercise 1*** : Show that $A \xrightarrow{1_A} A$ is an isomorphism.

From the definition, we need to find an inverse for $1_A$.

So: Need a map $A \xrightarrow{g} A$ such that

$$g \circ 1_A = 1_A \qquad \text{and} \qquad 1_A \circ g = 1_A$$

*Note*: The identity laws tell us that $g \circ 1_A = g$ and $1_A \circ g = g$.

So: Let $g = 1_A$! In other words, an inverse for $1_A$ is just $1_A$.

***Exercise 2*** : (*A map can have at most one inverse.*) Suppose $B \xrightarrow{g} A$ and $B \xrightarrow{k} A$ are both inverses for $A \xrightarrow{f} B$. Show that $g = k$.

If $g$, $k$ are inverses for $f$, then

$$g \circ f = 1_A = k \circ f \qquad \text{and} \qquad f \circ g = 1_B = f \circ k$$

So:

| | |
|---|---|
| $g = 1_A \circ g$ | *identity law* |
| $= (k \circ f) \circ g$ | *given* |
| $= k \circ (f \circ g)$ | *associative law* |
| $= k \circ 1_B$ | *given* |
| $\overset{\checkmark}{=} k$ | *identity law* |

***Note***: Our first proof in the game of category theory. We want to establish that $g = k$. We do this by linking $g$ with $k$ by means of a series of justified steps. The only things we can use are what's given to us in the problem (indicated by "*given*" on the right), and the game rules (*i.e.*, the rules for category theory).

***Notation:*** The inverse of $A \xrightarrow{f} B$ is denoted $f^{-1}$.

***Exercise 3a*** : (*If a map f has an inverse, then f can be cancelled on the left.*)

*Claim*: Suppose $A \xrightarrow{f} B$ has an inverse. Then for any $h$, $k$ with codomain $A$, if $f \circ h = f \circ k$, then $h = k$.

*Proof*: First let's identify what we're given. We're given a map $f$ with an inverse $f^{-1}$. This means we're given

$$f^{-1} \circ f = 1_A \qquad \text{and} \qquad f \circ f^{-1} = 1_B$$

We're also given two maps $h$, $k$ with codomain $A$ such that

$$f \circ h = f \circ k$$

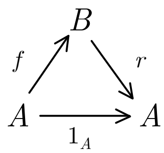
Given this, we want to now show that $h = k$. So begin with $h$ and see if we can get $k$ from what we're given and the rules of category theory:

| | |
|---|---|
| $h = 1_A \circ h$ | *identity law* |
| $= (f^{-1} \circ f) \circ h$ | *given* |
| $= f^{-1} \circ (f \circ h)$ | *associative law* |
| $= f^{-1} \circ (f \circ k)$ | *given* |
| $= (f^{-1} \circ f) \circ k$ | *associative law* |
| $= 1_A \circ k$ | *given* |
| $\overset{\checkmark}{=} k$ | *identity law* |

***Note***: As in *Ex.* #2, the tricky part here is the first step. There are two versions of the identity law ("left-handed" and "right-handed"), so be careful to use the right one.

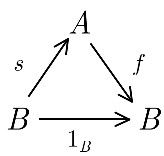
## 2. Retractions and Sections ("left-inverses", "right-inverses")

***Definition***. A ***retraction*** for a map $A \xrightarrow{f} B$ is a map $B \xrightarrow{r} A$ for which $r \circ f = 1_A$.




$r \circ f = 1_A$ $r$ is a "*left-inverse*" for $f$

***Definition***. A ***section*** for a map $A \xrightarrow{f} B$ is a map $B \xrightarrow{s} A$ for which $f \circ s = 1_B$.



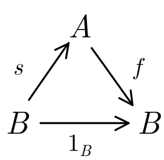

$f \circ s = 1_B$ $s$ is a "*right-inverse*" for $f$

Proposition 1 now tells us that if a map $f$ has a *section*, then we can do "*right-handed*" division with it.
Proposition 1* tells us that if a map $f$ has a *retraction*, then we can do "*left-handed*" division with it.

***Prop. 1***. If a map $A \xrightarrow{f} B$ has a *section*, then for any object $T$ and any map $T \xrightarrow{y} B$, there exists a map $T \xrightarrow{x} A$ for which $f \circ x = y$.

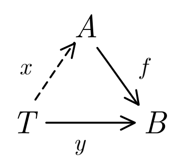
In other words:

*If we have*




$f \circ s = 1_B$

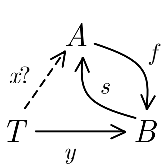
*then we can find $x$ in*




$f \circ x = y$

> ***Analogy with numbers:***
> Suppose we have a right-inverse for 4, call it ¼. This means 4 × ¼ = 1. We can now do "right-handed" division with 4. We can solve problems like 4 × $x$ = 5. What is $x$? It's 5 divided by 4. How do we compute this? If we have the right-inverse ¼, we compute it as $x$ = 5 × ¼.

*Proof*: We can summarize what we're given in the following diagram:




$f \circ s = 1_B$

*Strategy*: We want to show that an $x$ exists that "fills in" the dotted line in the diagram so that $f \circ x = y$. In other words, first doing this $x$, and then doing $f$, should be the same as just doing $y$. Look at the diagram and ask, How can we get from $T$ to $A$ (which is what $x$ should do) using just compositions of $s$, $y$, and/or $f$?

Let: $x = s \circ y$.

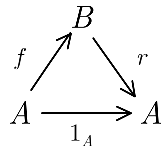
Now: Show that $x$, so-defined, satifies $f \circ x = y$.

$$\begin{aligned} f \circ x &= f \circ (s \circ y) && \textit{assumption} \\ &= (f \circ s) \circ y && \textit{associative law} \\ &= 1_B \circ y && \textit{given} \\ &\overset{\checkmark}{=} y && \textit{identity law} \end{aligned}$$

***Prop. 1****. If a map $A \xrightarrow{f} B$ has a *retraction*, then for any object $T$ and any map $A \xrightarrow{g} T$ , there exists a map $B \xrightarrow{t} T$ for which $t \circ f = g$.

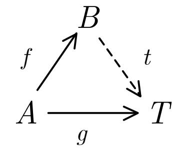
In other words:

*If we have*




$r \circ f = 1_A$

*then we can find t in*




$t \circ f = g$

<u>*Proof*</u>: Homework. *Very* similar to the proof for Prop. 1.

Proposition 2 tells us that a map $f$ with a *retraction* is "*left-cancellable*": if $f \circ x_1 = f \circ x_2$, for any $x_1$, $x_2$, then $x_1 = x_2$.
Proposition 2* tells us that if a map $f$ with a *section* is "*right-cancellable*": if $t_1 \circ f = t_2 \circ f$, for any $t_1$, $t_2$, then $t_1 = t_2$.

***Prop. 2***. If a map $A \xrightarrow{f} B$ has a *retraction*, then for any object $T$ and any maps $T \xrightarrow{x_1} A$, $T \xrightarrow{x_1} A$ , if $f \circ x_1 = f \circ x_2$, then $x_1 = x_2$.

<u>*Proof*</u>: Suppose $f$ has a retraction $r$. So $r \circ f = 1_A$. Then:

| | |
|---|---|
| $x_1 = 1_A \circ x_1$ | *identity law* |
| $= (r \circ f) \circ x_1$ | *given* |
| $= r \circ (f \circ x_1)$ | *associative law* |
| $= r \circ (f \circ x_2)$ | *given* |
| $= (r \circ f) \circ x_2$ | *associative law* |
| $= 1_A \circ x_2$ | *given* |
| ✓ $= x_2$ | *identity law* |

***Prop. 2****. If a map $A \xrightarrow{f} B$ has a *section*, then for any object $T$ and any maps $B \xrightarrow{t_1} T$, $B \xrightarrow{t_2} T$, if $t_1 \circ f = t_2 \circ f$, then $t_1 = t_2$.

<u>*Proof*</u>: Homework. *Very* similar to the proof for Prop. 2.

***Definition***. A ***monic*** is a *left-cancellable* map. In other words, a monic is a map $A \xrightarrow{f} B$ with the property that, for any object $T$ and any maps $T \xrightarrow{x_1} A$, $T \xrightarrow{x_1} A$, if $f \circ x_1 = f \circ x_2$, then $x_1 = x_2$.

***Definition***. An ***epic*** is a *right-cancellable* map. In other words, an epic is a map $A \xrightarrow{f} B$ with the property that, for any object $T$ and any maps $B \xrightarrow{t_1} T$, $B \xrightarrow{t_2} T$, if $t_1 \circ f = t_2 \circ f$, then $t_1 = t_2$.

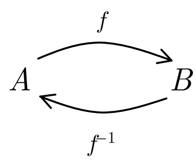
***ASIDE***: By Props 2, 2*, if a map has a retraction, then it's a monic; and if a map has a section, then it's an epic. But the converse isn't always true. If you're a monic, it's not necessarily the case that you have a retraction, and if you're a section, it's not necessarily the case that you're an epic. (See homework assignment.)

***Theorem***. If a map $A \xrightarrow{f} B$ has both a retraction $r$ and a section $s$, then $r = s$.

*Proof*: Suppose $f$ has a retraction $r$ and a seciton $s$. So $r \circ f = 1_A$ and $f \circ s = 1_B$. Then:

| | |
|---|---|
| $r = r \circ 1_B$ | *identity law* |
| $= r \circ (f \circ s)$ | *given* |
| $= (r \circ f) \circ s$ | *associative law* |
| $= 1_A \circ s$ | *given* |
| $\overset{\checkmark}{=} s$ | *identity law* |

***New Improved Definition***. A map $A \xrightarrow{f} B$ is an ***isomorphism*** if there exists another map $f^{-1}$ which is *both* a retraction and a section of $f$. Such a map $f^{-1}$ is called *the* inverse of $f$.




$$f \circ f^{-1} = 1_B$$
$$f^{-1} \circ f = 1_A$$

***Definition***. An endomorphism that is also an isomorphism is called an ***automorphism***.