11: Category Theory

Topics
I. Category of Finite Sets
II. Endomap, Identity Map, Composite Map
III. Definition of a Category
IV. Points of a set

Theory of ***objects*** and ***maps*** (*relations* between objects).

A ***category*** consists of a type of object and an associated map (plus a couple other things).

The Category $\mathcal{S}$ of Finite Sets

1. An ***object*** in this category = a finite set.

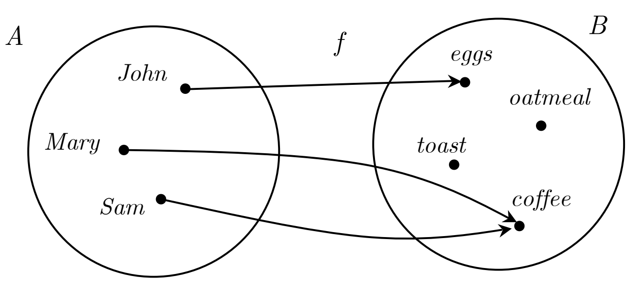
   <u>*exs.*</u> - set of all students in this class
   - set of all desks in this class
   - set of all letters in the alphabet

2. A ***map*** f in this category consists of the following:
   (a) A *set* A, called the <u>***domain***</u> of f.
   (b) A set B, called the <u>***codomain***</u> of f.
   (c) A <u>***rule***</u> for f that assigns to each element of A *exactly one* element of B.

the rule is also called a function, or a morphism, or simply an arrow

<u>***Example***</u>: Let $A = \{John, Mary, Sam\}$
$B = \{eggs, oatmeal, toast, coffee\}$
$f =$ "*favorite breakfast*" map

<u>***"Internal" diagram***</u>



<u>***"External" diagram***</u>

$$A \xrightarrow{f} B$$

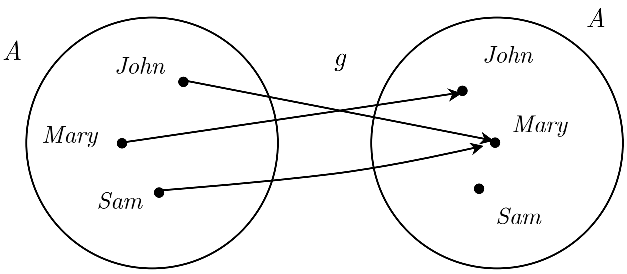
<u>The rule for f can also be represented by the following:</u>
$f(John) = eggs$
$f(Mary) = coffee$
$f(Sam) = coffee$

<u>***Essential Characteristic of an $\mathcal{S}$-map:***</u>
Every dot in domain must have exactly one arrow leaving and arriving at some dot in codomain.

Endomap = map in which domain and codomain are the *same* object

example:

Internal diagram



External diagram

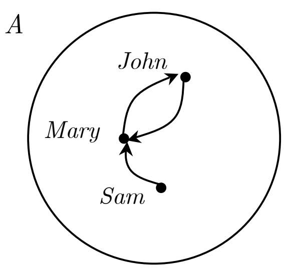
$$A \xrightarrow{g} A$$

g = "*favorite person*" map

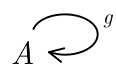
Check: Is g a legitimate map?

OR: More simply

Internal diagram



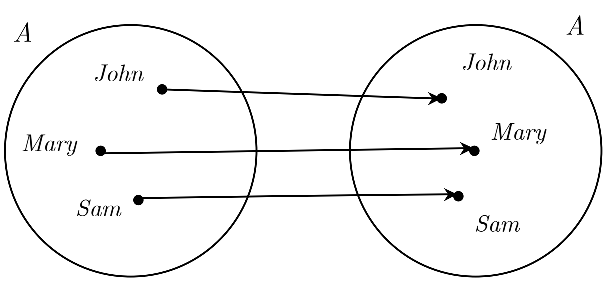
External diagram



Identity map = an endomap 1_A for which $1_A(a) = a$ for every element a in the domain A.

example:

Internal diagram

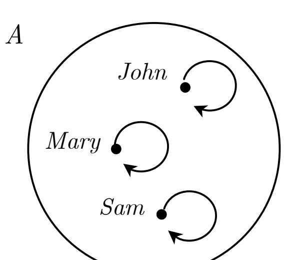


External diagram

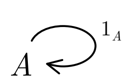
$$A \xrightarrow{1_A} A$$

OR:

Internal diagram

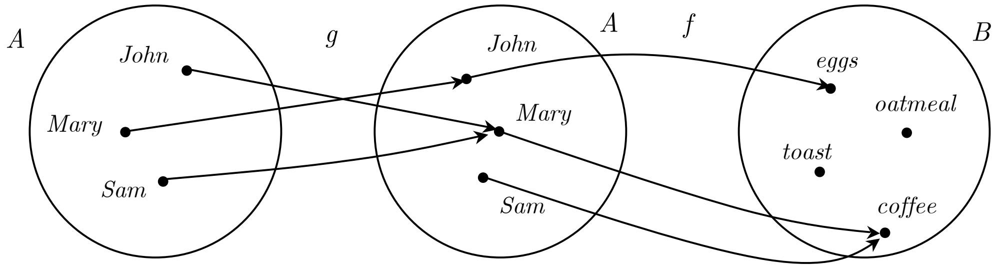


External diagram



Composition of maps : *Idea* - from any two maps f, g, for which the *domain* of one is the *codomain* of the other, you can construct a third map $f \circ g$ (a "shortcut").

Internal diagram



$g =$ "*favorite person*" map $f =$ "*favorite breakfast*" map

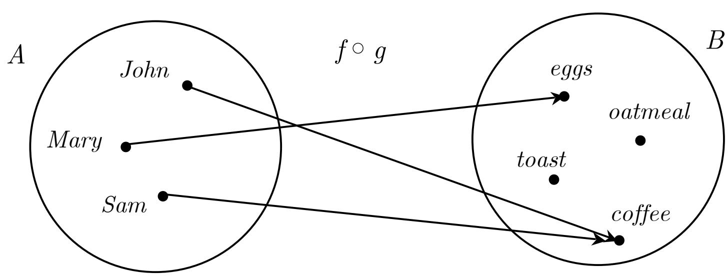
External diagram

codomain of $g = A =$ *domain of* f

$$A \xrightarrow{g} A \xrightarrow{f} B$$

First doing g and then f takes us from A to A, and then to B. Why not go *directly* from A to B?

Internal diagram



Rule for $f \circ g$ map:

$(f \circ g)(John) = coffee$

$(f \circ g)(Mary) = eggs$

$(f \circ g)(Sam) = coffee$

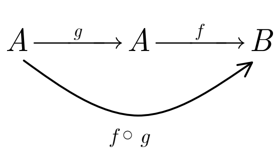
$f \circ g =$ "*favorite breakfast*" of "*favorite person*" map

NOTE! $f \circ g$ is the map you get by *first* doing g and *then* f.

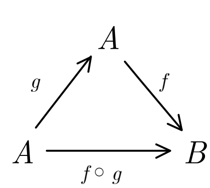
External diagram

$$A \xrightarrow{f \circ g} B$$

Can combine both of the above external diagrams:



OR



Now: *The official definition of a category....*

A category consists of:

(1) *Objects* $A,\ B,\ C,\ \ldots$

(2) *Maps* $A \xrightarrow{f} B, B \xrightarrow{g} C, \ldots$

This means that to fully specify a map in a general category, you need an arrow, its domain, and its codomain. (In the category $\boldsymbol{\mathcal{S}}$ *of finite sets, the arrow is given by a rule.)*

(3) *Identity maps* One per object: $A \xrightarrow{1_A} A, B \xrightarrow{1_B} B\ldots$

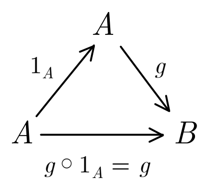
(4) *Composite maps* One for each pair of maps of the form $A \xrightarrow{f} B \xrightarrow{g} C$

Denoted $A \xrightarrow{f \circ g} C$

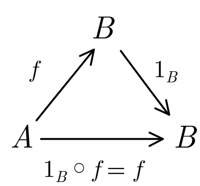
Rules for a category:

(1) *Identity Laws*

(a) For any map g with *domain* A, $g \circ 1_A = g$.



(b) For any map f with *codomain* B, $1_B \circ f = f$.

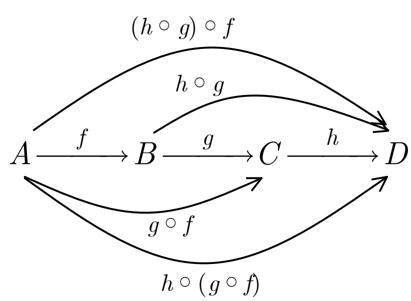


ASIDE: Identity maps are a bit like the number 1 in that "multiplying" ("composing") an identity map with another map yields that map. But note the restrictions: *Which* identity map to use for a given map, and *where* to place it (*left* or *right* of that map), depends on the domain or codomain of the given map!

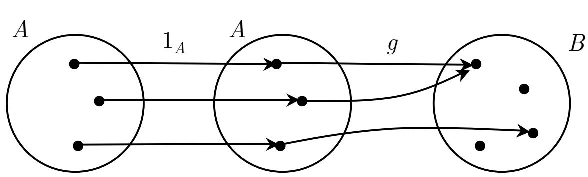
(2) *Associative Law*

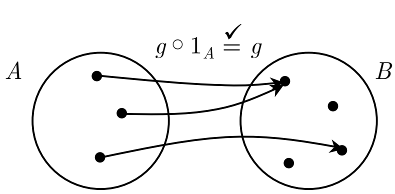
If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$

then $h \circ (g \circ f) = (h \circ g) \circ f$.



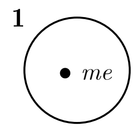
Quick check of Identity Law (a):



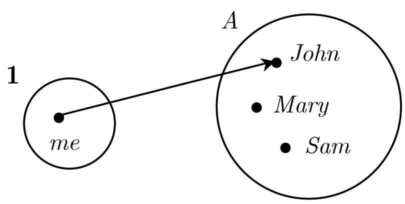


Singleton set = A set with one element. Denote it by **1**.

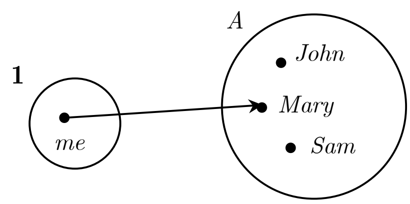
<u>ex.</u>



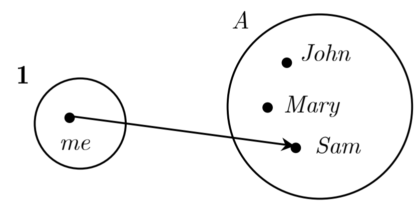
NOW: Consider maps from 1 to the set $A = \{John, Mary, Sam\}$



"*John*" map



"*Mary*" map

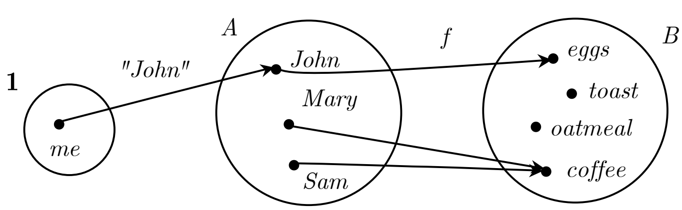


"*Sam*" map

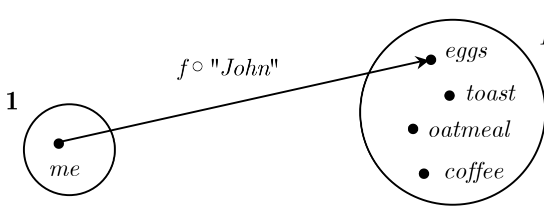
A ***point*** of a set X is a map $\mathbf{1} \to X$.

SO: The "*John*" map is a point of the set $A = \{John, Mary, Sam\}$.

<u>Claim</u>: The composition of a point and any other map is another point.



"*John*" is a point of A



$f \circ$ "*John*" is a point of B