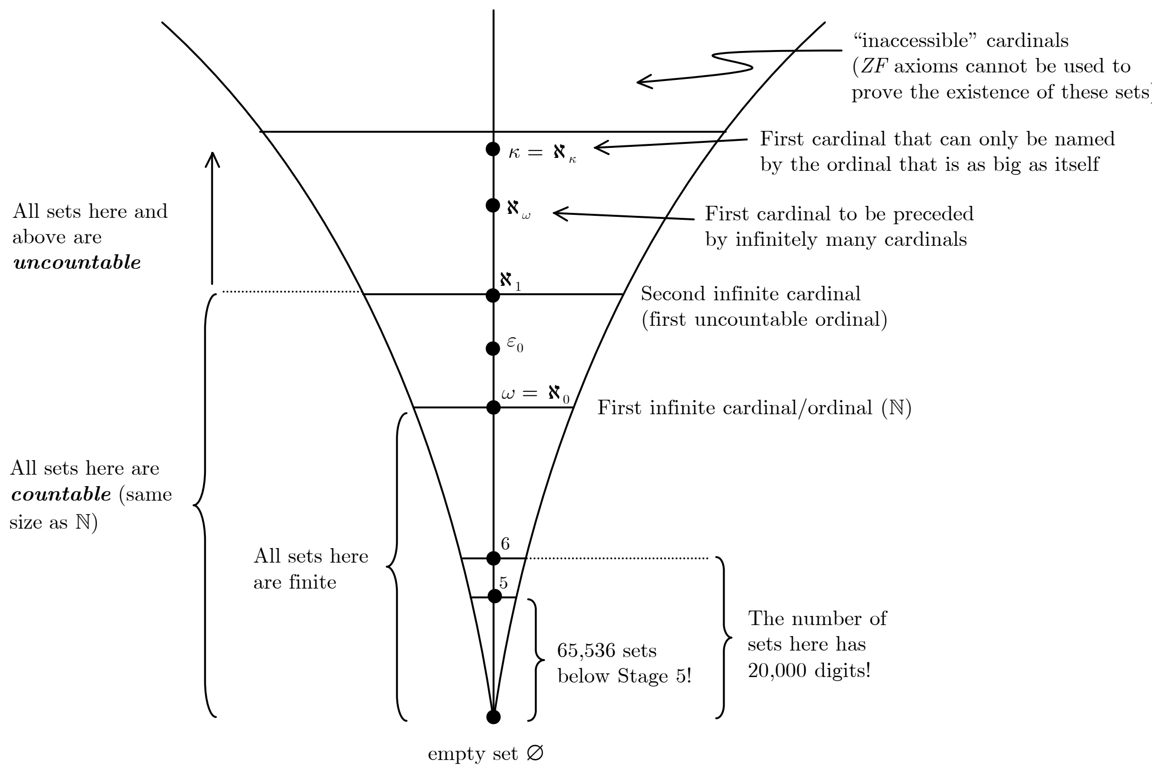
9: Problems with ZF Set Theory: The Skolem Paradox

Recall: *ZF* Set Theory -- attempt to mathematically codify the concept of infinity

I. Advantages of *ZF* Set Theory:

(A) Precise notion of "set": avoids paradoxes of the One and the Many (Russell's paradox, Set of Sets Paradox, *etc*)

(B) Precise notions of infinity: *ZF* Set theory includes Cantor's theory of ordinals and cardinals:



II. The Skolem Paradox

Problem of how to interpret ZF. Do we know what we're talking about? (Do we really know what *ZF* is *about*?)

***First note*:**

"Ordinary" languages: Interpretations are (usually) easy to fix. The speaker can always *point* to the objects being referred to in the language (*ostensive definitions*).

ZF Set Theory language: We can't point to pure sets. Is there a way to fix the subject matter of the language of *ZF* to *unambiguously* be about *sets*? The Löwenheim-Skolem Theorem says "No":

Löwenheim-Skolem Theorem:

If a *first order* formal system S is consistent, then S has a model whose domain is a ***countable*** set.

Applies to ZF Set Theory: ZF is a first-order formal system.

model: *an interpretation that makes all theorems of S true*

domain: *the set of Things that the interpretation is talking about*

UPSHOT: If S is consistent, then we can always interpret it as describing only ***countably*** many Things.

Consequences for ZF:

(1) *Problematic*: No matter how many true statements from the language of *ZF* we are given, we could never tell if the speaker was talking about sets or natural numbers (or *any* ***countable*** collection of Things).

(2) *Worse*: *What about **uncountable** sets?*

[Under the "intended" interpretation of *ZF*, there are such Things as uncountable sets (the Vast majority of sets).] *and* [*L-S* Theorem says *ZF* can be interpreted as being about only countable Things.]

How can an uncountable Thing be interpreted as a countable Thing?

The Skolem Paradox

Under its intended interpretation, *ZF* refers to uncountable sets.

BUT: The *L-S* Theorem says we can always interpret *ZF* as only referring to countable sets.

SO: ***How can we interpret an uncountable set in terms of a countable set?***

> ***Example***: How can we interpret the sentence "The powerset $\wp(\mathbb{N})$ of $\mathbb{N}$ is uncountable" *only* in terms of countable sets?

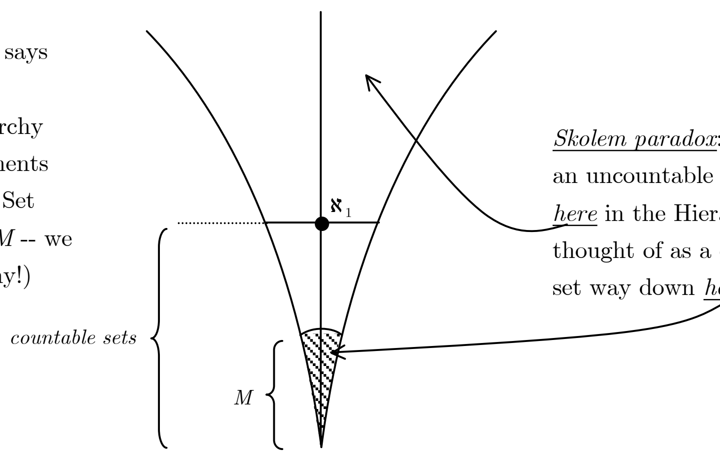
General idea: The *L-S* Theorem allows us to do the following:

Take a small slice *M* off the bottom of the Set Hierarchy such that:

(1) *M* is a countably infinite set, whose members are themselves countable sets.

(2) *M* serves as an interpretation of *ZF*: The members of *M* can be interpreted as the subject matter of *ZF*. Under this interpretation, an "*M*-set" corresponds to a "*ZF*-set".

L-S Theorem: *Everything ZF* says about all sets (countable and uncountable) in the Set Hierarchy can be reinterpreted as statements about *M*-sets. (*i.e.*, to do *ZF* Set Theory, all we really need is *M* -- we don't need the entire Hierarchy!)



Skolem paradox: How can an uncountable set way up *here* in the Hierarchy be thought of as a countable *M*-set way down *here*?

Formal Resolution of Skolem Paradox:

Recall: To say "Set *A* is uncountable" means "There is another set *B* such that the members of *A* cannot be *paired in 1-1 fashion* with the members of *B*".

under the "intended" interpretation, B is $\mathbb{N}$

... and this just means "*Another* set *C* exists whose members are the pairs of *A* and *B* members"

SO: Statements about uncountable sets are interpreted in *M* as statements about whether or not certain *M*-sets exist.

> ***Example***: The statement "The powerset $\wp(\mathbb{N})$ of $\mathbb{N}$ is uncountable" is interpreted in *M* as a statement about certain *M*-sets: "There is an *M*-set M_1 (corresponding to $\wp(\mathbb{N})$) and there is an *M*-set M_2 (corresponding to $\mathbb{N}$) and there is not an *M*-set corresponding to the set of pairs of members of M_1 and M_2"
>
> *i.e.*, "*Within M*, there is a set M_1 that looks like $\wp(\mathbb{N})$ and another M_2 that looks like $\mathbb{N}$, and these can't be paired."

Outside of M, we can see that all M-sets are really only countable. The M-set M_1 that M says is $\wp(\mathbb{N})$ really isn't: outside M, M_1 and $\mathbb{N}$ can be paired, but this requires the existence of a "pairing" set that isn't in M.

Lingering Conceptual Problems:

The *L-S* Theorem says there is nothing intrinsic to *ZF* that can determine what its intended interpretation is. In particular: Anything you can do in *ZF*, you can do in *M*. But *we* know that *ZF* extends to Things outside *M* (*i.e.*, it extends to sets in the full Set Hierarchy).

BUT: How do we know that what we take to be the full Hierarchy *really* is the intended interpretation of *ZF*? What if what we think is the full hierarchy is *really* a small slice, call it *M'*, near the bottom of an even larger hierarchy?

In particular: What we think are ***uncountable*** sets in our hierarchy may *really* be ***countable*** *M'*-sets in the larger hierarchy.

Suggests a relativism of the following sort (Skolem):

> *A set can only be said to be countable or uncountable* ***relative*** *to an interpretation of ZF.*

But recall: The distinction between countable and uncountable sets is the basic distinction between types of infinity:

Countably infinite sets: $\mathbb{N}$, ω, $\aleph_0$ -- "first level" of infinity

Uncountable sets: $\aleph_1$, $\aleph_2$, $\aleph_3$, ... -- each labels the next higher level of infinity

Are we thus left with a relative concept of infinity?