08. Zermelo-Fraenkel (*ZF*) Formal Set Theory

Motivation: To translate Naive Set Theory into a *formal system* (see lecture on *Rigorization and Proof*)

Primitives of ZF: Individuals (infinite): sets ("pure" iterative sets)
One Property: set-membership (denoted by "$\in$")

Formal Rules of ZF: First-Order Logic, *ZF* axioms

How to play the game of set theory

ZF axioms (8):

(ZF1) ***Axiom of Extension.*** Two sets x, y are the same if and only if they have the same members.

$$x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)$$

(ZF2) ***Empty Set Axiom.*** A set x exists that has no members. ← *This tells us that at least one set exists*

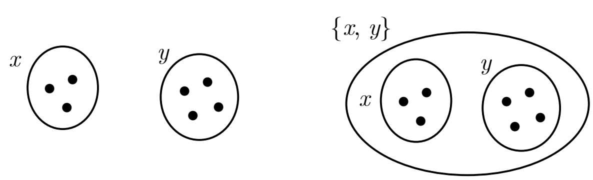
$$(\exists x)(\forall y){\sim}(y \in x)$$

Notation: (ZF1) and (ZF2) entail there is a unique empty set: Call it $\varnothing$.

(ZF3) ***Pairing Axiom.*** Given any sets x and y, there is a "pair" set z whose members are x and y.

$$(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow (w = x \vee w = y))$$

Notation: (ZF1) and (ZF3) entail there is a unique pair set for any given x, y: Call it $\{x, y\}$.
Speical Case: The ***singleton set*** $\{x\}$ is the special case $\{x, x\}$.



(ZF4) ***Union Axiom.*** Given any set x, there is a "union" set y which has as its members all members of members of x.

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists w)(w \in x \,\&\, z \in w))$$

Notation: (ZF1) and (ZF4) entail there is a unique union for any set x: Call it $\bigcup x$.
Let $x \cup y$ represent the union set $\bigcup\{x, y\}$ of the pair set of x and y.

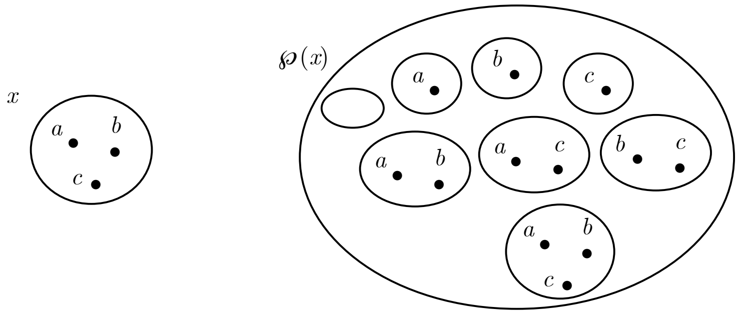
$\{x, y\}$ $\qquad$ $\bigcup\{x, y\}$ *or* $x \cup y$

(ZF5) ***Powerset Axoim.*** Given any set x, there is a set y which has as its members all sets whose members are also members of x (*i.e.*, y contains all the "subsets" of x).

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall w)(w \in z \to w \in x))$$

Notation: (ZF1) and (ZF4) entail there is a unique powerset for any set x: Call it $\wp(x)$.
Define $z \subseteq x$ ("z is a subset of x") as $\forall w(w \in z \to w \in x)$. Then (ZF5) can be written as:

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \subseteq x)$$



Def. 1: For any set x, the ***successor*** of x is the set $x' = x \cup \{x\}$.

(ZF6) ***Axiom of Infinity.*** A set x exists that contains $\varnothing$, and the successor of each of its elements.

$$(\exists x)(\varnothing \in x \;\&\; (\forall y)(y \in x \to y' \in x))$$

Terminology: A set that satisfies (ZF6) is called a ***successor set***.
Comment: This axiom guarantees the existence of a set x such that $\varnothing$ is a member of x, and for any set y, if y is a member of x, then so is its successor $y' = y \cup \{y\}$.
By construction, $\varnothing$ is in x.

Then: So is $\varnothing \cup \{\varnothing\}$, or $\{\varnothing\}$.
Then: So is $\{\varnothing\} \cup \{\{\varnothing\}\}$ or $\{\varnothing, \{\varnothing\}\}$.
Then: So is $\{\varnothing, \{\varnothing\}\} \cup \{\{\varnothing, \{\varnothing\}\}\}$ or $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$.
Etc...
So: The "minimal" successor set is given by $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots\}$

Let's call this set ω, and call each of it members by the following:

$\varnothing = 0$
$\{\varnothing\} = 1 = \{0\}$
$\{\varnothing, \{\varnothing\}\} = 2 = \{0, 1\}$
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = 3 = \{0, 1, 2\}$
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\} = 4 = \{0, 1, 2, 3\}$
etc., ...

} *The natural numbers!*

Note: $1 = 0'$, $2 = 1' = 0''$, $3 = 2' = 0'''$, *etc.*

<u>*SO*</u>: $\omega = \{0, 1, 2, 3, \ldots\}$

Can now form:

ω', call it $\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, 3, \ldots, \omega\}$

ω'', call it $\omega + 2 = \omega \cup \{\omega'\} = \{0, 1, 2, 3, \ldots \omega, \omega + 1\}$

etc....

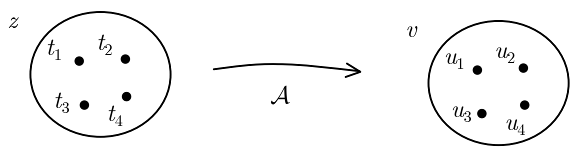
What about "higher order" ordinals? Need the following axiom:

(ZF7) ***<u>Replacement Axiom Scheme</u>.*** Given a relation $\mathcal{A}(x, y)$ that relates every set x to a unique set y, then for any set z, we can form a new set v which has as its members all the sets that are related to members of z under $\mathcal{A}$.

$$(\forall x)(\exists! y)\mathcal{A}(x, y) \rightarrow (\forall z)(\exists v)(\forall u)(u \in v \leftrightarrow (\exists t)(t \in z \;\&\; \mathcal{A}(t, u)))$$

<u>ASIDE</u>: The symbol "∃!" here means "there exists a unique..."

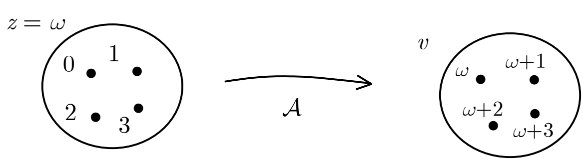
<u>*Comment*</u>: The members of v are formed by collecting together all the sets to which the members of z are mapped by $\mathcal{A}$. You start with the set z and get the set v by replacing all the members of z with their counterparts under the relation $\mathcal{A}$. This is called an “Axiom Scheme” since it holds for all possible relations $\mathcal{A}$ (so there’s really one axiom per relation $\mathcal{A}$: you can build a new set from an original by using any appropriate available function).



$\mathcal{A}(t_1, u_1)$ holds,
$\mathcal{A}(t_2, u_2)$ holds,
etc...

<u>example</u>: Let $\mathcal{A}(x, y)$ be the relation that holds just when $x \in \omega$ and $y = \omega + x$ (*i.e.*, $\mathcal{A}(x, y)$ holds just when y is the *xth successor* of ω).

<u>*Then*</u>: By (ZF7), there is a set v such that for every $t \in \omega$, $\omega + t \in v$.



$\mathcal{A}(0, \omega)$ holds,
$\mathcal{A}(1, \omega{+}1)$ holds,
etc...

<u>*Now*</u>: Let $\omega \times 2 = \omega \cup v = \{0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$

<u>*And*</u>: Construct $\omega \times 3$ using (ZF7) with the relation $\mathcal{A}'(x, y)$ that holds just when $x \in \omega$ and $y = \omega \times 2 + x$ (y is the *xth* successor of $\omega \times 2$). Similarly for $\omega \times 4$, $\omega \times 5$, *etc...*

<u>*Now*</u>: Let $\mathcal{A}''(x, y)$ be the relation that holds just when $x \in \omega$ and $y = \omega \times x$ (y is the *xth multiple* of ω). This generates sets of the form $\omega^2, \omega^3, \omega^4$, *etc.*, ...

<u>**ex.**</u>

$$\omega^2 = \left\{ \begin{matrix} 0, & 1, & 2, & \cdots \\ \omega, & \omega + 1, & \omega + 2, & \cdots \\ \omega \times 2, & (\omega \times 2) + 1, & (\omega \times 2) + 2, & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \right\}$$

Can continue in similar manner to construct ever increasing hierarchy of ordinals!

(ZF8) ***Axiom of Foundation***. Every non-empty set x contains a member that has no members in common with x.

$$(\forall x)(\sim x = \varnothing \rightarrow (\exists y)(y \in x \;\&\; \sim(\exists z)(z \in y \;\&\; z \in x)))$$

Comment: This axiom says that for any set x other than the empty set, there is a "minimal" member y of x that has no members in common with members of x. This rules out circular chains of sets ($e.g.$, $x \in y$ and $y \in z$ and $z \in x$) and infinitely descending chains of sets. In particular, it rules out the possibility of a set being a member of itself:

Lemma: For any set x, $x \notin x$.

ex: $x = \{x\}$, $x = \{x, u, v, ...\}$

Proof: Suppose there's a set x such that $x \in x$.

Then: There's a "pair" set $z = \{x\}$. (ZF3, degenerate pair $\{x, x\}$.)

Now: z must have a "minimal" member y such that $z \cap y = \varnothing$. (ZF8)

But: The *only* member of z is x, and $z \cap x \neq \varnothing$. (Since $x \in x$, $x \in z$.)

Recall: Russell's Paradox

The Russell "set" R is defined by: $x \in R \leftrightarrow x \notin x$. Is R a member of itself? If $R \in R$, then $R \notin R$, and if $R \notin R$, then $R \in R$.

BUT: The above *lemma* entails that R *cannot be a set!*
WHY? If R is a set, then it must be the set of all sets (since the *lemma* states that *all* sets are sets that are not members of themselves). But if R is the set of all sets, it must contain itself. So R cannot be a set. (So what is R? It's a "collection" of sets that is not itself a set.)

II. Natural Number Arithmetic

General Claim: Natural number arithemtic can be reduced to ZF set theory.

Def. 2: A set F is ***hereditary with respect to successor*** if for any set x, if x is a member of F, then so is its successor x'.

Def. 3: A set x is a ***natural number*** if, for every set F, if $\varnothing \in F$ and F is hereditary with respect to successor, then $x \in F$.

Def. 4 (Addition): For any $m \in \mathbb{N}$,

(i) $m + 0 = m$

(ii) for any $n \in \mathbb{N}$, $m + n' = (m + n)'$

ex.
$3 + 2 = 3 + (0')'$
$= (3 + 0')'$ (Def. 4ii)
$= ((3 + 0)')'$ (Def. 4ii)
$= (3')'$ (Def. 4i)
$= 4' = 5$

Def. 5 (Multiplication): For any $m \in \mathbb{N}$,

(i) $m \times 0 = 0$

(ii) for any $n \in \mathbb{N}$, $m \times n' = (m \times n) + m$

ex.
$3 \times 2 = 3 \times (0')'$
$= (3 \times 0') + 3$ (Def. 5ii)
$= ((3 \times 0) + 3) + 3$ (Def. 5ii)
$= (0 + 3) + 3$ (Def. 5i)
$= 6$

Can now show that the "Peano" axioms for first order arithmetic hold in ZF:

Peano Axioms for "First Order" Arithmetic

(PA1) 0 is a natural number.

(PA2) For each natural number x, x' is a natural number.

(PA3) For all natural numbers x, $0 \neq x'$.

(PA4) For all natural numbers x and y, if $x' = y'$, then $x = y$.

(PA5) For any set F of natural numbers containing 0, if F is hereditary with respect to successor, then F contains all natural numbers. ← *Principle of Weak Mathematical Induction*

Claim: From $\mathbb{N}$ one can generate $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ (integers, rationals, reals, complex numbers).

Additional Axiom for ZF:

(AC) ***Axiom of Choice.*** For any non-empty set x, there is a set y which has precisely one element in common with each member of x.

Comment: *AC* doesn't tell you *how* to construct y; *i.e.*, it doesn't say what the "choice" function is that you use to pick out the members of y from members of x. All the other axioms *do* give you recipes for the construction of new sets. For this reason, the status of *AC* as an axiom is sometimes debated. It is needed in order to prove that all sets can be well-ordered, so it's important for the theory of ordinal and cardinal numbers.