07. Cantor's Theory of Ordinal and Cardinal Numbers

Topics
I. Ordinals
II. Iterative Conception of a Set
III. Ordinals as Sets
IV. Cardinal Numbers
V. Transfinite Arithmetic and the Continuum Hypothesis

Ordinals - measure the "length" or "shape" of a set

Cardinals - measure the "size" of a set (in correlation sense)

I. Ordinal Numbers

Definition: A ***well-ordering*** of a set A (finite or infinite) is an imposition of order on the members of A that
(1) singles out one member as the first (unless X is the empty set)
(2) for each member or set of members already specified, singles out its successor (unless no members are left).

Examples:

1. $< \ldots, -2, -1, 0, 1, 2, \ldots >$ whole numbers
   - *Not a well-ordering*: Doesn't specify a first.

2. $< 0, \ldots, ¼, \ldots, ½, \ldots, 1, \ldots, 1½, \ldots, 2, \ldots >$ non-negative rational numbers
   - *Not a well-ordering*: Doesn't specify a successor of 0.

3. $< \underbrace{0, 1, 2, 3, \ldots,}_{\textit{naturals}} \underbrace{\ldots, -3, -2, -1}_{\textit{negative wholes}} >$ *whole numbers*
   - *Not a well-ordering*: Doesn't specify a successor to ℕ.

4. $< 0, 1, 2, 3, \ldots >$ natural numbers ℕ
   - *Well-ordering.*

   no successor specified at the end; but this is okay, since no members are left.

5. $< 1, 2, 3, \ldots., 0 >$ naturals ℕ
   - *Well-ordering.* (0 is specified as the successor to all the non-negative naturals.)

6. $< \overbrace{0, 2, 4, 6, \ldots,}^{\textit{even naturals}} \overbrace{1, 3, 5, \ldots}^{\textit{odd naturals}} >$ naturals ℕ

   some still left (i.e., odds)

   successor to all evens

   none left -- no need for successor

   - *Well-ordering.*

Note: Well-orderings of different sets may have the same "shape" ("length").

examples: $< 1, 2, 3, 4, ..., 0 >$ well-ordering of ℕ

$< 3, 5, 7, ..., 1 >$ well-ordering of odd naturals

same "shape"/"length"

$<1, 2, 4, 5, ..., 0, 3>$ well ordering of ℕ with "one more member"

slightly "longer"!

Ordinal numbers: measure "length" of well-ordered sets in correlation sense:

Defining Characteristics

(i) One ordinal is first.

(ii) For each ordinal, there is another which is its successor.

(iii) For each set of ordinals (finite or infinite), there is an ordinal that succeeds them all.

- First ordinals are the natural numbers 0, 1, 2, 3, ...

measure the "length" of finite sets (for finite sets, "length" and "size" are similar concepts)

- The ordinal that succeeds these is called "ω"

- *length of* $< 0, 1, 2, 3, 4, ... >$
- *Not a natural number!*

- Next ordinal is called "$\omega + 1$"

just a name at this point; not a sum

Next is called "$\omega + 2$"

etc...

length of $< 1, 2, 3, 4, ..., 0 >$

length of $< 1, 2, 4, 5, ..., 0, 3 >$

- The ordinal that succeeds all "ω + _" ordinals is called "$\omega + \omega$" or "$\omega \times 2$"

length of $< 0, 2, 4, ..., 1, 3, 5, ... >$

- Next is "$(\omega \times 2) + 1$"

etc...

length of $< 2, 4, 6, ..., 1, 3, 5, ..., 0 >$

A small part of the ordinals:

0, 1, 2, 3, ...

ω, $\omega + 1$, $\omega + 2$, ...

$\omega \times 2$, $(\omega \times 2) + 1$, $(\omega \times 2) + 2$, ...

$\omega \times 3$

$\vdots$

ω^2, $\omega^2 + 1$, $\omega^2 + 2$, ...

ω^3

$\vdots$

ω^ω, $\omega^\omega + 1$, $\omega^\omega + 2$, ...

$\vdots$

ω^{ω^ω}, ..., $\omega^{\omega^{\omega^{w}}}$, ...

First ordinal to succeed all of these is called "ε_0".

Claim: There are as many ordinals preceding ω as there are preceding ε_0.

II. The Iterative Conception of a Set

What are "pure" sets? (Recall, these are supposed to be the objects of set theory: sets whose members themselves are sets)

Standard approach: Construction metaphor. To specify what "pure" sets are, we will construct them in stages. From our basic concepts, we know of at least one *unique* "pure" set: the empty set $\varnothing$. So why not construct our "universe" of pure sets based on $\varnothing$?

the primordial purest of pure sets!
the Garden of Eden set!

Iterative Construction of Pure Sets:

Rule A: Construct the empty set $\varnothing$.

Rule B: At any stage n, construct all possible sets (that haven't yet been constructed) whose members are taken only from prior stages.

Stage 1: First set is the empty set: $\varnothing$

Stage 2: Only one set can be constructed: $\{\varnothing\}$

Stage 3: Two sets can be constructed: $\{\{\varnothing\}\}$ and $\{\varnothing, \{\varnothing\}\}$

Stage 4: Twelve sets can be constructed:

2 new single-member sets:	$\{\{\{\varnothing\}\}\}$
	$\{\{\varnothing, \{\varnothing\}\}\}$
5 new two-member sets:	$\{\varnothing, \{\{\varnothing\}\}\}$
	$\{\varnothing, \{\varnothing, \{\varnothing\}\}\}$
	$\{\{\varnothing\}, \{\{\varnothing\}\}\}$
	$\{\{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
	$\{\{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$
4 new three-member sets:	$\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$
	$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
	$\{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$
	$\{\{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$
1 new four-member set:	$\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

In general, if number of sets constructed by stage n is k, then number of sets constructed by stage $n + 1$ is 2^k.

SO: Stage 4 has $2^4 = 16$ sets total.

Stage 5 has $2^{16} = 65, 636$ sets

Stage 6 has $2^{65,636}$ sets!

etc...

a huge universe of pure sets!

For each ordinal α, there is a Stage α!

Key features of "construction" metaphor:

(1) No end to set construction: Hierarchy of sets has no top!
(2) Members of a set "exist" before the set itself. (Recall: members themselves are sets)
(3) *Potential* infinity of sets: Hierarchy is infinite in the sense that, for any Stage α, we can always progress to the next Stage $\alpha + 1$.

Can the set hierarchy be thought of as actually infinite?

Is the "construction" metaphor just a manner of speaking about sets?

Note: If the hierarchy is actually infinite, then there might seem to be a problem with the set of all sets.
If the hierarchy exists as a complete whole, then we can consider the set that contains all sets in it.
But this set of all sets both is and is not a member of itself!

Cantor's-Way-Out: The set of all sets is an "inconsistent totality"; a "misbehaving set.

III. Ordinals as Sets

Recall: Ordinals are measures of the "length" of sets:

ω is the length of {0, 1, 2, 3, ...}

$\omega + 1$ is the length of {0, 1, 2, 3, ..., ω}

etc.

Note: Natural numbers are ordinals.

SO: 5 is the length of {0, 1, 2, 3, 4}

1 is the length of {0}

0 is the length of $\varnothing$

Motivates following identification:

$0 = \varnothing$

$1 = \{\varnothing\}$

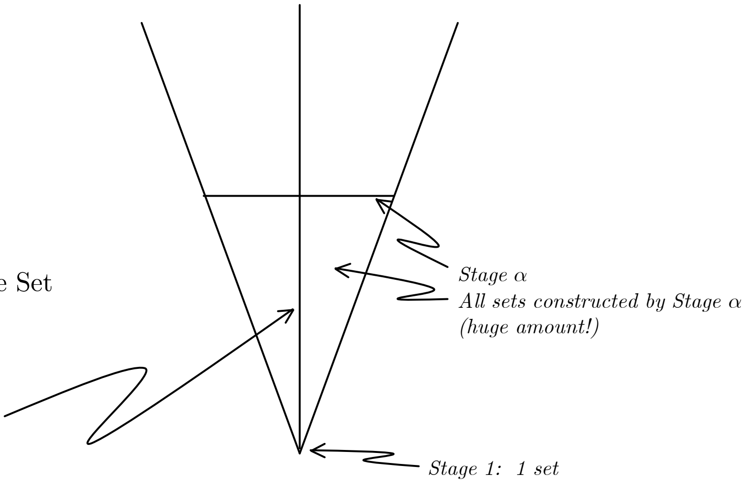
$2 = \{\varnothing, \{\varnothing\}\}$

$3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

$\vdots$

$\omega = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots\}$

SO: Ordinals can be considered as sets -- part of the Set Hierarchy



Recall: *Ordinals label the stages of set construction. Represent them as the "back-bone" of the V-shaped Set Heirarchy*

IV. Cardinal Numbers

Motivation: No limit to how big an infinite set can be.

BUT: There *is* a limit to how *small* it can be.

example: All infinite sets are at least as big as $\mathbb{N}$ (think of $\mathbb{N}$ as the first infinite size).

Terminology: A ***countable*** set is any set that is either finite or the same size as $\mathbb{N}$.

An ***uncountable*** set is any set bigger than $\mathbb{N}$.

Cardinal numbers measure the "size" of sets. *They are the smallest ordinals of a given set size.*

- The *finite cardinals* are the natural numbers -- measure the size of finite sets.

example: The size of the set of planets,
{*Mercury*, *Venus*, *Earth*, *Mars*, *Saturn*, *Jupiter*, *Uranus*, *Neptune*}
is the Cardinal 8.

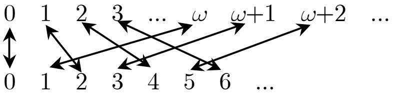
- The first *infinite cardinal* is called $\aleph_0$ ("aleph null") and measures the size of $\mathbb{N}$.

All ordinals between ω and ε_0 are countable: they *all* have the same size as $\mathbb{N}$ (but different "lengths").

ω is the smallest ordinal of this size.

example: *Claim*: ω has the same size as $\omega + \omega$

Proof: Pair first ω ordinals in $\omega + \omega$ onto even members of ω.
Pair second ω ordinals in $\omega + \omega$ onto odd members of ω.



SO: $\mathbb{N}$ and ω and $\aleph_0$ all name the *same* set; namely, the set of natural numbers.

ω measures its "length".

$\aleph_0$ measures its "size" -- tells us how many members it has.

- The next infinite cardinal is called $\aleph_1$ and measures the size of the ordinal that succeeds all countable ordinals (must exist!).

Between $\aleph_0$ and $\aleph_1$ there are many ordinals.

$\omega + 1, \omega + 2, ...$

$\omega \times 2, ...$

$\omega^\omega ...$

All have the same size $\aleph_0$, but all have different lengths.

- Cardinals are labeled by ordinals:

$$\underbrace{\aleph_0, \aleph_1, \aleph_2, ..., \aleph_\omega, \aleph_{\omega+1}, ..., \aleph_{\varepsilon_0}, ..., \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}}, ...$$

All of these are cardinals that come much after their labels -- *eg.*, $\aleph_0$ comes much after 0

first cardinal to succeed all of these is labeled by the ordinal that it is:

$\kappa = \aleph_\kappa$

Incredibly big! So big that it needs itself to say how big it is!

V. Transfinite Arithmetic and the Continuum Hypothesis

Some results:

(1) If κ and λ are cardinals, at least one of which is infinite, and $\kappa \geq 1$, then

$$\kappa + \lambda = \lambda + \kappa = \kappa \times \lambda = \lambda \times \kappa = \kappa$$

example: $\aleph_7 + \aleph_5 = \aleph_7 \times \aleph_5 = \aleph_7$

(2) If κ is a cardinal, finite or infinite, then $\kappa < 2^\kappa$

Motivation: If set A has n members, then $\wp(A)$ has 2^n members.

example: $A = \{a, b, c\}$

$\wp(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}. \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Result (2) then follows from the claim that $\wp(A)$ is always bigger than A, for any A.

Particular Consequence of Result (2):

$$\underset{\substack{\nearrow \\ \textit{size of } \mathbb{N}}}{\aleph_0} < \underset{\substack{\nwarrow \\ \textit{size of } \wp(\mathbb{N})}}{2^{\aleph_0}}$$

Further Claim: $\mathbb{R}$ and $\wp(\mathbb{N})$ have the same size.

SO: $\aleph_0 < 2^{\aleph_0}$ says that $\mathbb{R}$ is bigger than $\mathbb{N}$.

Recall ***Cantor's Unanwered Question***: How much bigger than $\mathbb{N}$ is $\mathbb{R}$?

The "Continuum Hypothesis" (*CH*) states that $\mathbb{R}$ is the "next infinite size up" from $\mathbb{N}$. More precisely:

$$\boxed{CH\text{: } 2^{\aleph_0} = \aleph_1}$$

Why it's called the "Continuum" Hypothesis:

The points on a line form a "***continuum***", which has two basic properties:

1. Between any two points is another (*denseness* property).
2. There are no "gaps" between points.

The rationals are *dense*: between any two rationals is another.

BUT: There are gaps in the rationals (where the irrationals would be).

The reals include both rationals and irrationals: they "fill in the gaps" between the rationals.

So — measures the "size" of a continuum: measures how "close" together the points on a line are.

And *CH* claims that this "closeness" is given by $\aleph_1$.

Is *CH* true?

Technical results: Godel (1938): Can't prove that *CH* is false within formal (ZF) set theory.

Cohen (1963): Can't prove that *CH* is true within formal (ZF) set theory.