06. Naive Set Theory

Topics
I. Sets and Paradoxes of the Infinitely Big
II. Naive Set Theory
III. Cantor and Diagonal Arguments

I. Sets and Paradoxes of the Infinitely Big

Recall: Paradox of the Even Numbers

Claim: There are just as many even natural numbers as natural numbers

natural numbers = non-negative whole numbers (0, 1, 2, ..)

Proof: $\{0, 1, 2, 3, 4, \ldots\ldots\ldots, n, \ldots\ldots\}$

$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \; \updownarrow \qquad \updownarrow$

$\{0, 2, 4, 6, 8, \ldots\ldots\ldots, 2n, \ldots\ldots\}$

In general: 2 criteria for comparing sizes of sets:

(1) *Correlation criterion*: Can members of one set be paired with members of the other?

(2) *Subset criterion*: Do members of one set belong to the other?

Can now say:

(a) There are as many even naturals as naturals in the *correlation* sense.

(b) There are less even naturals than naturals in the *subset* sense.

Moral: (To talk about the infinitely big, just need to be clear about what's meant by size) ← *notion of sets makes this clear*

Bolzano (1781-1848)

Promoted idea that notion of infinity was fundamentally *set-theoretic:*

(To say something is infinite is just to say there is some set with infinite members)

"God is infinite in knowledge"
means
"The set of truths known by God has infinitely many members"

SO: Are there *infinite* sets? ← *"a many thought of as a one" -Cantor*

Bolzano: *Claim*: The set of truths is infinite.

Proof: Let p_1 be a truth (ex: "Plato was Greek")

Let p_2 be the truth "p_1 is a truth".

Let p_3 be the truth "p_3 is a truth".

In general, let p_n be the truth "$p_{n\text{-}1}$ is a truth", for any natural number n.

Dedekind: (1831-1916) *Claim*: The set of thoughts is infinite.

Proof: Let s_1 be a thought.

Let s_2 be the thought that s_1 is a thought.

Let s_3 be the thought that s_3 is a thought.

In general, let s_n be the thought that $s_{n\text{-}1}$ is a thought, for any natural number n.

BUT: Can these sets *really* be treated as complete wholes?

Russell's Paradox

Let R be the set of all sets that do not belong to themselves.

Claim: R belongs to itself *if and only if* R does not belong to itself.

Proof: (i) Suppose R belongs to R.
Then R is a set that does not belong to itself.
So R does not belong to R.
(ii) Suppose R does not belong to R.
Then R is a set that belongs to itself.
So R does belong to R.

Without any constraints on what counts as a set...
There should be sets that ***do not*** *belong to themselves:*
- *The set of all cats*
- *The set* $\{0, 1, 2, 3\}$
- *The set* $\{0, 1, 2, 3, \{0, 1\}\}$
- *The set* $x = \{a, b, c\}$

... and there should be sets that ***do*** *belong to themselves:*
- *The set of sets*
- *The set of all non-cats*
- *The set* $y = \{a, b, c, y\}$

Russell's Paradox is a paradox of the One and the Many: It looks like R can't be thought of as a "one".
SO: Sets were introduced initially (in part) to address paradoxes of the Infinitely Big. But now it seems we've just replaced them with paradoxes of the One and the Many!

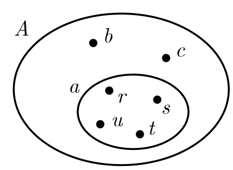
II. Naive Set Theory: Basics

First: We want our theory to codify the *essential properties* of sets, and *no more*.
So: Restrict attention *only* to sets, and *nothing more*. (*i.e.*, "pure" sets, whose members themselves are sets)

1. Membership relation: *"is a member of"*, denoted by $\in$.

Notation: "$x \in y$" means "*x is a member of y*".

example: $A = \{a, b, c\}, \quad a = \{r, s, t, u\}$
$a \in A, \quad r \in a$



2. Principle of Extension: Two sets are equal if they have the same members.

Or: $x = y$ *if and only if* (*iff*) for all sets z, $(z \in x) \Leftrightarrow (z \in y)$

example: $A = \{a, b, c\}, \quad B = \{b, c, a\}$
$A = B$

Note: One result of this is that the two sets a and $\{a\}$ are *not* the same. The first set a may have any number of members. The second set $\{a\}$ has only one member (namely, a).

3. Empty Set: There is only one set that has no members. Call it the *empty set* $\varnothing = \{\}$.

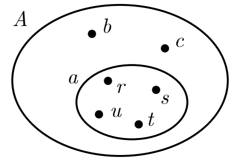
Proof: Suppose x and y are distinct sets with no members.
Then: x and y have the same members (*i.e.*, none).
So: $x = y$, by the Principle of Extension.

Note: Again the two sets $\varnothing$ and $\{\varnothing\}$ are distinct. $\varnothing$ has no members, whereas $\{\varnothing\}$ has one member, namely $\varnothing$!

4. Subset relation: x is a *subset* of y just when every member of x is a member of y.

Notation: "$x \subseteq y$" means "*x is a subset of y*"

So: $x \subseteq y$ *iff* for all sets z, $(z \in x) \Rightarrow (z \in y)$



$A \subseteq A, \quad a \subseteq A$

Notes:
1. $\varnothing$ is a subset of every set
2. For any set x, $x \subseteq x$
3. $x = y$ *iff* $x \subseteq y$ and $y \subseteq x$

5. Power Set: The *power set* $\wp(x)$ of x is the set of all subsets of x.

examples: $C = \{a\}, \quad \wp(C) = \{\varnothing, \{a\}\}$

$B = \{a, b\}, \quad \wp(B) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

$A = \{a, b, c\}, \quad \wp(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

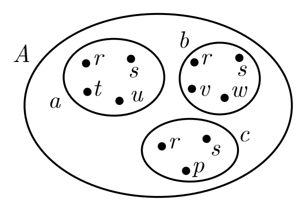
Note: If a is the empty set, then it's a subset of C. But if a is not the empty set (if it contains at least one member), then it is *not* a subset of C! Because then it's not true that every member of a is a member of C. C only contains one member; whereas a may contain many.

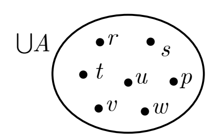
6. Union Set: The *union set* $\bigcup x$ of x is the set consisting of sets that are members of the members of x.

Or: $\bigcup x = \{$all sets z such that $z \in y$ for some $y \in x\}$
$= \{$all sets z such that there is a set y such that, $y \in x$ *and* $z \in y\}$

example: $A = \{a, b, c\}, \quad a = \{r, s, t, u\}, \; b = \{r, s, v, w\}, \; c = \{r, s, p\}$

$\bigcup A = \{r, s, t, u, v, w, p\}$





Special Case: Union of a pair set $\{x_1, x_2\}$.

notation: $\bigcup\{x_1, x_2\}$ is sometimes written $x_1 \cup x_2$.

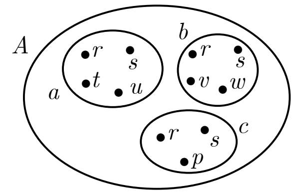
$\bigcup\{x_1, x_2\} = x_1 \cup x_2 = \{$all z such that $z \in x_1$ or $z \in x_2\}$

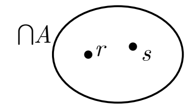
7. Intersection Set: The *intersection set* $\bigcap x$ of x is the set consisting of all sets that are members of every member of x.

Or: $\bigcap x = \{$all sets z such that $z \in y$ for all $y \in x\}$
$= \{$all sets z such that for every set y, *if* $y \in x$, *then* $z \in y\}$

example: $A = \{a, b, c\}, \quad a = \{r, s, t, u\}, \; b = \{r, s, v, w\}, \; c = \{r, s, p\}$

$\bigcap A = \{r, s\}$





Special Case: Intersection of a pair set $\{x_1, x_2\}$.

notation: $\bigcap\{x_1, x_2\}$ is sometimes written $x_1 \cap x_2$.

$\bigcap\{x_1, x_2\} = x_1 \cap x_2 = \{$all z such that $z \in x_1$ and $z \in x_2\}$

Now Consider:

$\bigcup\varnothing = \varnothing$

$\bigcap\varnothing = ?$ ← *every set!*

<u>*WHY*</u>? Let z be some arbitrary set.

<u>*Then*</u>: It's (vacuously) true that z belongs to every member of $\varnothing$ (since there are no members of $\varnothing$).

<u>*So*</u>: *Every* set should be in $\bigcap\varnothing$!

$$\bigcap\varnothing = \{\text{all sets } z \text{ such that for every set } y, \underbrace{\textit{if } y \in \varnothing, \textit{ then } z \in y}_{\text{This "if...then" sentence is vacuously true!}}\} = \{\text{all sets}\}$$

<u>*BUT*</u>: We *don't* want to allow for a set that contains all sets (Russell's Paradox!).

<u>*So*</u>: Need a restriction:

> ***Restriction***
> $\bigcap x$ is defined *only* if $x \neq \varnothing$.

<u>*Thus:*</u> *We want to always have union sets, but not necessarily intersection sets (see axioms later on).*

III. Cantor and the Size of Sets

- adopted correlation criterion for set-size:

(1) Set A has the same size as set B just when members of A can be paired with members of B.

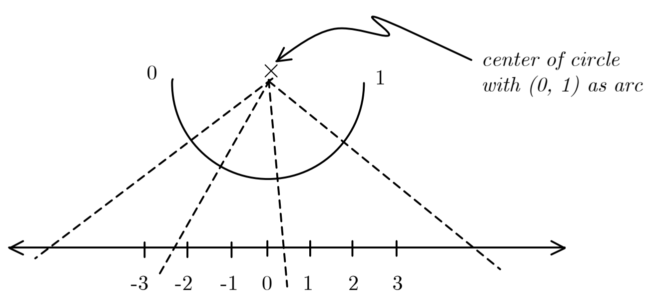
(2) Set A is bigger than set B just when all members of B can be paired with some members of A, but not with all of them.

Some results:

1. There are as many even natural numbers as natural numbers.

2. There are as many real numbers between 0 and 1 as there are real numbers.

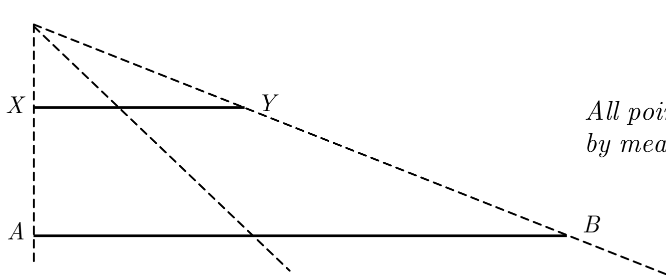
<u>*Proof:*</u>



Every point on the arc (i.e., real number between 0 and 1) is paired with a real number on the real number line by means of the dashed projection lines that originate at the circle's center.

3. All line segments have the same number of points.

<u>*Proof:*</u>



All points in $\overline{XY}$ are paired with all points in $\overline{AB}$ by means of dashed projection lines.

4. ***There are more real numbers between 0 and 1 than there are natural numbers.***

Proof: (**"Diagonal" Argument**)

(i) Pair natural numbers with decimal expansions of reals between 0 and 1. There are many ways to do this. One particular way is the following:

0 ↔	0 . 3 3 3 3 ...	$= 1/3$	*all reals between 0 and 1 can be given an infinite decimal expansion*
1 ↔	0 . 1 4 1 5 ...	$= \pi - 3$	
2 ↔	0 . 4 1 4 2 ...	$= \sqrt{2} - 1$	
3 ↔	0 . 5 0 0 0 ...	$= 1/2$	
⋮	⋮		

(ii) Construct a real between 0 and 1 that is not listed in the table:

(a) Go down the "diagonal" of the table starting at the first digit in the decimal expansion of the first real.

(b) Write "3" if the digit in the diagonal is a 4; write "4" if the digit in the diagonal is anything else.

our example: 0.4334...

(iii) This real is not listed in the table!

By construction, it differs from the first real in its first decimal place; it differs from the second real in its second decimal place, etc. In general, it differs from all listed reals (no matter how they are listed).

BUT: The table contains all the natural numbers (in its first column).

SO: There are more real numbers between 0 and 1 than there are natural numbers.

Recall: There are just as many reals between 0 and 1 as there are reals.

SO: There are more real numbers than there are natural numbers (even though both are infinite).

$\mathbb{R}$ = set of real numbers

$\mathbb{N}$ = set of natural numbers

5. ***There are more sets of natural numbers than there are natural numbers.***

Proof: (**"Diagonal" Argument**)

(i) Pair natural numbers with sets of natural numbers.

Represent a set of natural numbers as a sequence of "yes's" and "no's" depending on whether successive naturals belong to it or not.

Examples:

Represent $\{0, 2, 4, 6, 8, ... \}$ as <yes, no, yes, no, yes, ... >

Represent $\{1, 2\}$ as <no, yes, yes, no, no, no, no, ... >

SO: One way to do Step (i) is the following:

	0	1	2	3 ...
0 ↔	yes	no	yes	...
1 ↔	no	no	yes	...
2 ↔	no	no	no	...
3 ↔				
⋮	⋮	⋮	⋮	

(ii) Construct a set of natural numbers that is not listed in the table in the following way:
Go down the diagonal. Write "no" for each "yes", and "yes" for each "no".

our example: <no, yes, yes, ... >

(iii) By construction, this set of naturals is not listed in the table: It differs from the first listed set in its first member; it differs from the second listed set in its second member, etc. It differs from all listed sets of naturals, no matter how they are listed.
BUT: The table lists all natural numbers (in its first column).
SO: There are more sets of natural numbers than there are naturals.

Recall: The powerset $\wp(A)$ of a set A is the set of all subsets of A.

SO: Result #5 can be stated as: $\wp(\mathbb{N})$ is larger than $\mathbb{N}$.

Further Claim: For any set A, $\wp(A)$ is larger than A.

Consequence: No limit to how large an infinite set can be!
- $\mathbb{N}$ is infinite.
- $\wp(\mathbb{N})$ is larger.
- $\wp(\wp(\mathbb{N}))$ is larger still, *etc...*

One Big Question ("Cantor's Unanswered Question"):

How much larger than $\mathbb{N}$ is $\mathbb{R}$?

(a) Is it the "next infinite size up" from $\mathbb{N}$?

(b) Are there intermediate sizes between $\mathbb{N}$ and $\mathbb{R}$?

The "***Continuum Hypothesis***" is the claim that the answer is (a).