04. The Calculus

Topics

I. The Calculus and Infinitesimals

2 Sets of Problems

Properties of Curves
- slope of tangent to curve
- area under a curve

(*geometrical*)

Continuously-varying Quantities
- *example*: How do distance and speed vary under constant acceleration?

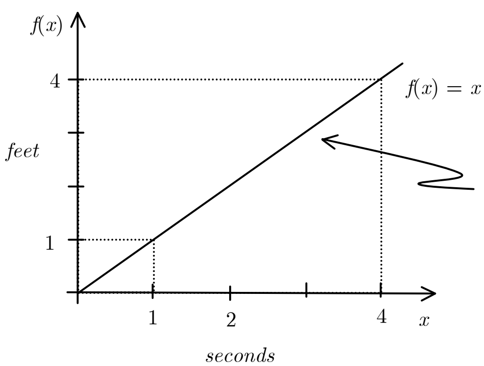
(*physical*)

Example 1: Slope of tangent to curve (Differential Calculus)

First: Constant speed motion:

Constant speed motion = motion for which the rate at which distance changes per time is constant.

Can represent geometrically by a curve on a graph with vertical axis labeled in units of distance and horizontal axis labeled in units of time. One example:



Curve $f(x) = x$ (*i.e.*, straight line) represents relation between time and distance for object moving at constant speed of x feet in x seconds.

slope = *rise/run* = *distance/time* = *speed*

⇨ For such a graph:

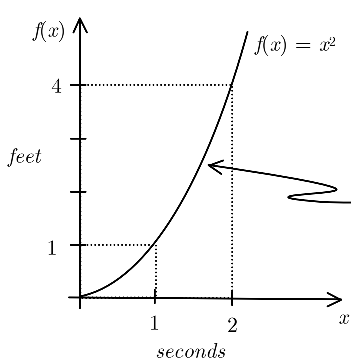
(***constant slope = constant speed***)

Now: Accelerated (non-constant speed) motion:

Accelerated motion = motion for which the rate at which distance changes per time (speed) is itself changing; *i.e.*, non-constant speed.

From above: (***non-constant speed = non-constant slope***)

One example:



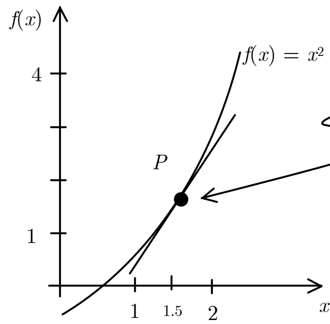
Curve $f(x) = x^2$ represents relation between time and distance for object moving at constant acceleration of x^2 feet in x seconds.

slope is non-constant: *rise/run* is not constant

(*i.e.*, object's speed is constantly changing between 0 seconds and 2 seconds)

Question: For accelerated motion, what is the object's speed at a given *instant*? (Say, at $x = 1.5$ seconds?)

Geometrically:

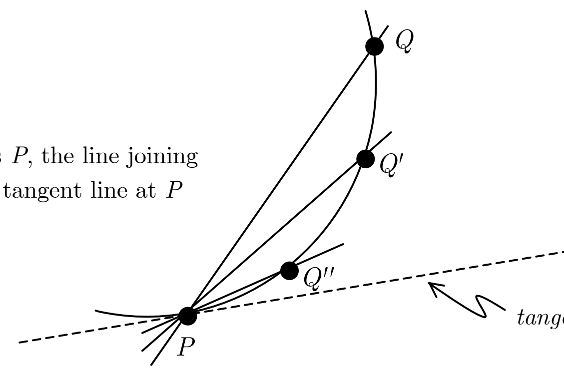


What is the slope (speed) of the *tangent line* to the curve $f(x) = x^2$ at $x = 1.5$ *sec*?

In general, how do you determine the slope of the tangent line to a curve at a point P?

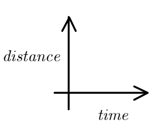
Claim: Tangent at P is the line joining P to a point Q that is *infinitely close* to P.

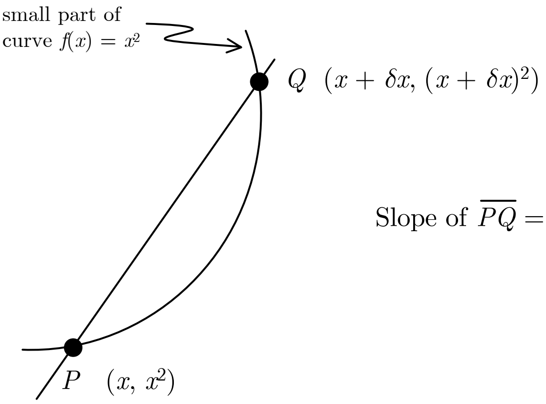
Idea: As Q approaches P, the line joining Q to P approaches the tangent line at P



Can now calculate the slope of the tangent to P, just from knowldege of slopes:

Assume Q is infinitely close to P and both are on curve $f(x) = x^2$:





Let δx be an *infinitely small* ("***infinitesimal***") quantity

$$\text{Slope of } \overline{PQ} = \frac{rise}{run} = \frac{f(x+\delta x) - f(x)}{(x + \delta x) - x} = \frac{(x+\delta x)^2 - x^2}{\delta x}$$

$$= \frac{x^2 + 2x\delta x + \delta x^2 - x^2}{\delta x}$$

$$= 2x + \delta x$$

infinitely small quantity: ***Ignore it***!

SO: Slope of $\overline{PQ} = 2x$

SO: "instantaneous" speed of object at 1.5 *sec* is $2 \times 1.5 = 3$ *ft*/*sec*

Terminology: The slope of the tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$ is called the ***derivative*** $f'(x_0)$ of $f(x)$ at x_0.

Coherent Reasoning?

What exactly is δx?

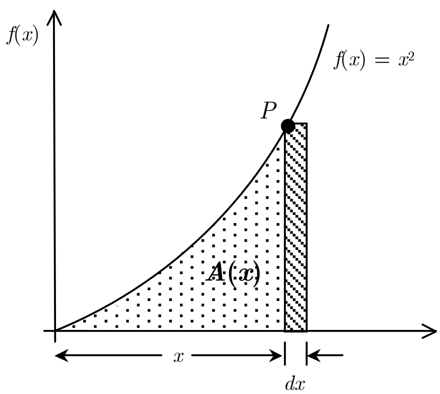
(a) Enough like 0 to be discarded?

(b) But can't be exactly 0!

Recall: Can't divide by 0:
If we could, then since $n \times 0 = m \times 0$ for any numbers n, m, we would have $n = m$ for any n, m.

Example 2: Area under a curve (Integral Calculus)

Find area $A(x)$ under curve $f(x) = x^2$.



Reasoning:

(An infinitesimal increase in x (call it δx, or just dx)) *leads to* (An infinitesimal increase in area $A(x)$ (call it dA) by an infinitely thin strip of height x^2 and width dx)

So: $dA = x^2\,dx$

Now: Consider $A(x)$ as the infinite sum of infinitely thin strips, each with infintely small area dA. Symbolically, we write:

$$A(x) = \int dA = \int x^2 dx$$

Terminology: $A(x)$ is called the ***indefinite integral*** of the function $f(x) = x^2$.

Coherent Reasoning? What is an infinite sum of infinitely small rectangles?

How to calculate an indefinite integral:

From $dA = x^2 dx$, we can write:

$$\frac{dA}{dx} = x^2$$

Divide increase in $A(x)$ by increase in x. Get x^2.

Can now use method of Example 1 "backwards": *There* we knew the ratio on the left and wanted the value on the right. *Here*, we have the value on the right and want the numerator on the left.

Terminology: $A(x)$ is also called the ***anti-derivative*** of $f(x) = x^2$.

The anti-derivative is the "inverse" of the derivative! Integration "undoes" differentiation.

$A(x)$ is the function with the property that, when you take its derivative, you get x^2.

II. The Status of Infinitesimals

Leibniz (1646-1716): "useful fictions"

Newton (1642-1727): "These ultimate ratios with which the quantities vanish are indeed not ratios of ultimate [*sc.* infinitesimal] quantities, but limits to which the ratios of quantities vanishing without limit always approach, to which they may come up more closely than by any given difference but beyond which they can never go." - Moore pg. 65

i.e., infinitesimals are not *actual*, but *potential.*

Berkeley (1685-1753):

The Calculus is incoherent:

The Analyst; or, A Discourse addressed to an Infidel mathematician. Wherein it is examined whether the Object, Principles, and Inferences of the modern Analysis are more distinctly conceived, or more evidently deduced, than Religious Mysteries and Points of Faith.

L' Hôpital (1661-1704:

Calculus textbook:

"A quantity which is increased or decreased by a quantity which is infinitely smaller than itself may be considered to have remained the same."

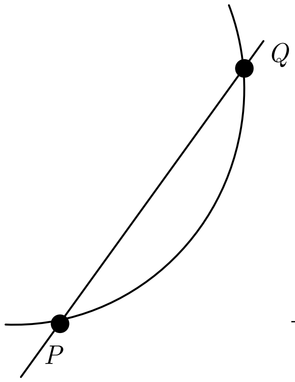
"A curve may be regarded as the totality of an infinity of straight segments, each infinitely small: or... as a polygon with an infinite number of sides. - Moore pg. 65

Cauchy (1789-1857)
Weierstrauss (1815-1897)
} provided *rigorous* foundations for the Calculus based on the concept of a ***limit***

think of as representing potential infinity, not actual; a (rigorous) way of talking that does not explicitly refer to infinitesimals.

III. The Concept of a Limit

Example: Slope of tangent to curve $f(x) = x^2$



$$\text{slope of } \overline{PQ} = \frac{2x\delta x + \delta x^2}{\delta x}$$

Define: Tangent to curve at P is the ***limit*** of all lines $\overline{PQ}$ as Q approaches P

This means:

(1) The smaller δx is, the closer $\dfrac{2x\delta x + \delta x^2}{\delta x}$ is to the slope of the tangent; *and*,

(2) You can get as *arbitrarily close* to the slope of the tangent as you care to specify:

> For any number ε no matter how small, you can always find a (finite!) number δx such that $\dfrac{2x\delta x + \delta x^2}{\delta x}$ lies within ε of the slope of the tangent.

Terminology: "$\displaystyle\lim_{\delta x \to 0} \frac{2x\delta x + \delta x^2}{\delta x} = 2x$" means "The limit as δx goes to zero of $\dfrac{2x\delta x + \delta x^2}{\delta x}$ is $2x$".

Two points:

(1) δx is *never* zero -- it's always a finite number.

(2) δx is *never* "fully" infinitesimally small -- it's always a finite number.

1. Rigorous definition of the limit of a function:

"Epsilon-Delta" Definition of a Limit (Cauchy)

Let $f(x)$ be defined in a neighborhood of x_0. Then,

$$\lim_{x \to x_0} f(x) = L$$

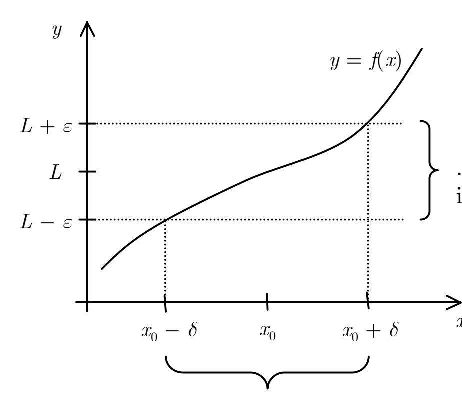
if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that,

if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

"whenever x gets within δ of x_0"

"$f(x)$ gets within ε of L"

"The limit as x goes to x_0 of $f(x)$ is given by the value L"



... then $f(x)$ is in this range

If x is in this range...

No matter how small ε is chosen, δ can be chosen small enough so that $f(x)$ is within ε of L. In other words, you can get as arbitrarily close to L as you like.

IV. Limits and Derivatives

Can now give rigorous definition of derivative.

Recall: The "derivative" $f'(x_0)$ of $f(x)$ at the point x_0 is the slope of the tangent line to the graph of $f(x)$ at the point $(x_0, f(x_0))$.

Rigorous definition of the derivative of a function at a point:

Let $f(x)$ be defined in a neighborhood of x_0. Then the derivative $f'(x_0)$ of $f(x)$ at x_0 is given by

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \longleftarrow \frac{\textit{change in rise}}{\textit{change in run}} \quad \textit{as } \Delta x \textit{ goes to zero}$$

Here Δx is a small finite quantity (never zero!) representing the change in run.

Note: This defines a function $f'(x)$ on all points x that the original function $f(x)$ is defined on and for which the limit exists. This function $f'(x)$ is called the ***derivative** of $f(x)$* and is also written as

$$f'(x) = \frac{d}{dx} f(x) \qquad \text{or just simply} \qquad f'(x) = \frac{df}{dx}$$

***ASIDE**: It still looks like we're dividing infinitely small quantities, now called "differentials" dx instead of "infinitesimals" δx. Namely, df/dx might be thought of as an infinitely small change in f, namely df, divided by an infinitely small change in x, namely dx. But these symbols can now simply be thought of as short-hand for the rigorous definition above.*

Properties of the derivative function f'(x): *(What you learn in Calc I)*

1. If c is a constant, then $\dfrac{d}{dx} cx^n = ncx^{n-1}$.

   examples: $\dfrac{d}{dx} 6x^2 = 12x$ $\qquad \dfrac{d}{dx} 3 = 0$

2. (*Sum Rule.*) If $f(x)$ and $g(x)$ are functions of x, then $\dfrac{d}{dx}(f+g) = \dfrac{df}{dx} + \dfrac{dg}{dx}$.

3. (*Product Rule.*) If $f(x)$ and $g(x)$ are functions of x, then $\dfrac{d}{dx}(fg) = \dfrac{df}{dx} g + f \dfrac{dg}{dx}$.

4. (*Chain Rule.*) If $f(g)$ is a function of g, and $g(x)$ is a function of x, then $\dfrac{df}{dx} = \dfrac{d}{dx} f(g(x)) = \dfrac{df}{dg}\dfrac{dg}{dx}$.

   example: Let $f(g) = g^2$, and $g(x) = 3x + 2$. Then $\dfrac{df}{dx} = \dfrac{df}{dg}\dfrac{dg}{dx} = (2g)(3) = 6g = 18x + 12$

Note: The rigorous definition of the derivative function lets us rigorously define the *anti-derivative* or *indefinite integral*: For a given function $f(x)$, if there exists a function $A(x)$ such that $dA/dx = f(x)$, then $A(x)$ is called the ***anti-derivative*** *of f(x)*, or the ***indefinite integral*** *of f(x)*, and is also written symbolically as

$$A(x) = \int f(x)dx$$

This definition means that integration is the "inverse" of differentiation in so far as

$$\frac{d}{dx}A(x) = \frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

examples: $\int x^2 dx = (\textit{the function whose derivative is } x^2) = \frac{1}{3}x^3$

$$\int (3x^3 + 5x)dx = \frac{3}{4}x^4 + \frac{5}{2}x^2$$

To explain the squiggly "S" notation (the idea that the integral represents a sum), we need to now consider the concept of an infinite sum.

V. Limits and Infinite Sums

The concept of a limit allows a precise definition of an *infinite sum.*

Note first: Addition is only defined for "finite" input.

In arithmetic, addition is simply a 2-place function; it takes two pieces of input and outputs a sum.

SO: The following "infinite" sum makes no sense without further ado:

$$S_\infty = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... \equiv \sum_{k=1}^{\infty} \frac{1}{2^k}$$

"The sum from $k = 1$ to ∞ of $1/2^k$"

BUT: We *can* calculate "***partial sums***": $S_n \equiv \sum_{k=1}^{n} \frac{1}{2^k}$

$$\left.\begin{aligned} S_1 &= 1/2 \\ S_2 &= 1/2 + 1/4 = 3/4 \\ S_3 &= (1/2 + 1/4) + 1/8 = 7/8 \\ S_4 &= ((1/2 + 1/4) + 1/8) + 1/16 = 15/16 \end{aligned}\right\}$$

all of these partial sums are finite sums: they only involve two pieces of input

Now form a *sequence* of all these partial sums:

$$\{1/2, 3/4, 7/8, 15/16, ...,\} \equiv \{S_n\}$$

where each member is given by the *function* $f(n) = \frac{2^n - 1}{2^n}$

We say the sequence $\{S_n\}$ is *generated* by the function $f(n)$.

Now define the *limit* of a sequence of partial sums:

Definition of the limit of a sequence

The sequence $\{S_n\}$ generated by the function $f(n)$ has a limit L if, for every $\varepsilon > 0$, there is an $N > 0$ such that

$$\text{if } n \geq N, \quad \text{then } \left| f(n) - L \right| < \varepsilon.$$

$\{S_n\}$ has a limit L if there's a point N in the sequence afterwhich $f(n)$ stays within ε of L, for any ε.

Write: $\lim\limits_{n\to\infty}\{S_n\} = L.$

"The limit as n goes to ∞ of the sequence $\{S_n\}$ is L."

Now define the ***infinite sum*** S_∞ as the limit as n goes to infinity of the sequence $\{S_n\}$ of its partial sums.

Rigourous definition of an infinite sum

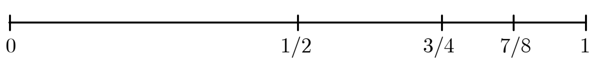
The infinite sum $\sum\limits_{k=1}^{\infty} g(k) \equiv S_\infty$ is the limit of the sequence of its partial sums,

$$S_\infty = \lim_{n\to\infty}\{S_n\}$$

if such a limit exists:

Can now address Zeno's Paradoxes:

Runner Paradox:



Racetrack: Starting line at 0 and finish-line at 1.

Claim: Achilles will never cross the finish-line.

Assumptions: (a) The track is infinitely divisible.

SO: It's length = an infinite sum of finite pieces

(b) An infinite sum of finite pieces is infinite.

Aristotle: Rejected (b). As an infinite sum of finite pieces, the racetrack is *potentially* infinite, and not *actually* infinite.

Calculus: Rejects (b). Some infinite sums are finite (depends on whether or not the sequence of their partial sums has a finite limit).

Runner case: $S_\infty = \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots \equiv \sum\limits_{k=1}^{\infty} \dfrac{1}{2^k}$

The sequence of partial sums $\{S_n\} = \{1/2, 3/4, 7/8, 15/16, ...,\}$is generated by the function

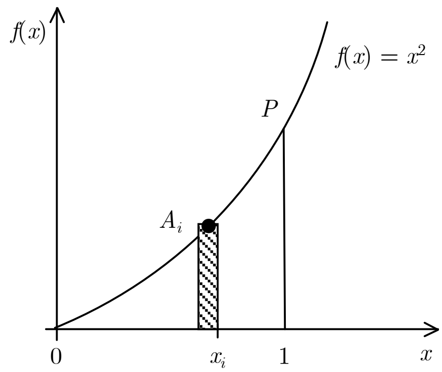
$$f(n) = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

So $\quad S_\infty = \lim\limits_{n\to\infty}\left(1 - \dfrac{1}{2^n}\right) = 1 \qquad$ (*i.e.*, as we expect, the racetrack's length is 1)

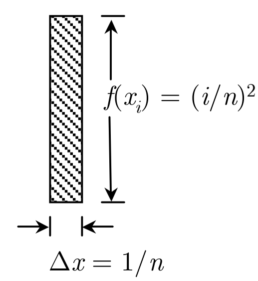
VI. Infinite Sums and Integrals (Optional)

Particular Example:

How to calculate the *definite* integral $\int_0^1 dA = \int_0^1 x^2 dx$ ← *the area under the curve $f(x) = x^2$ between the points $x = 0$ and $x = 1$.*



*i*th rectangle:



Area $= f(x_i)\Delta x$
$= (i/n)^2(1/n)$

- Divide interval [0, 1] into n segments of equal length $\Delta x = 1/n$.
- Construct n rectangles with these segments as bases.
- Identify the location of the ith rectangle as $x_i = i/n$ (the endpoint of the ith segment).

THEN: The height of the ith rectangle is $f(i/n) = (i/n)^2$.

AND: The area A_i of the ith rectangle is $A_i = (i/n)^2(1/n)$.

AND: The total area A = sum of all A_i as n gets very large (*i.e.*, as Δx goes to 0):

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}(i/n)^2(1/n) = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty}\frac{2+(3/n)+(1/n^2)}{6} = \frac{1}{3}$$

this step requires knowing that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

In general:

Rigorous Definition of the Definite Integral (Riemann)

Let $f(x)$ be a continuous function defined over the interval $[a, b]$. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

where $x_i = i(b - a)/n$ and $\Delta x = (b - a)/n$.

Precisely how the squiggly "S" symbol of integration represents an infinite sum, rigorously defined as the limit of a sequence of partial sums!

How the Definite Integral relates to the Indefinite Integral and Differentiation:

The Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function defined over the interval $[a, b]$. Let $A(x)$ be an anti-derivative of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)dx = A(b) - A(a)$$