03. Early Greeks & Aristotle

I. Early Greeks

1. Anaximander (b. 610 B.C.)

"to apeiron"
- "the unlimited", "unbounded"
- fundamental substance of reality
- underlying substratum for change
- neutral substratum in which opposites/strife are reconciled

Solution to the Problem of the One and the Many:
Observable objects = composites of the four elements: earth, air, fire, water.
Question: How do such opposing elements combine to form objects?
Answer: Through the mediation of *to apeiron*

2. The Pythagoreans (Pythagoras b. 570 B.C.)

the physical world = product of the imposition of *"peras"* (limits) on *"**a peiron**"*

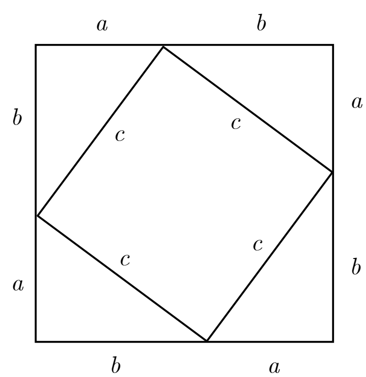
result = order/harmony

basis for this order = natural numbers

Pythagoras' Theorem

Claim: In any right triangle, the square of the hypotenuse is equal to the sum of the squares on the other two sides.

Proof:



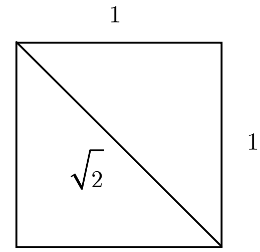
Consider two squares, one inscribed inside the other.

$$\text{area of outer square} = (a+b)^2 = c^2 + 4 \times (\text{area of } \Delta abc)$$
$$= c^2 + 4 \times (1/2\,ab)$$
$$= c^2 + 2ab$$

Or: $a^2 + 2ab + b^2 = c^2 + 2ab$

So: $a^2 + b^2 = c^2$

Irrationality of $\sqrt{2}$



Pythagoras' Theorem for length of diagonal of a unit square:

$$1^2 + 1^2 = \left(\sqrt{2}\right)^2$$

Claim: The square root of 2 is not a rational number.

Proof: Suppose $\sqrt{2}$ is a rational number.

- *Then*: There are integers p, q such that $\sqrt{2} = p/q$.
- *Or*: $2 = p^2/q^2$.
- *Suppose*: p/q is in *lowest terms*.
- *Now*: $2q^2 = p^2$.
- *So*: p^2 must be even. Hence p must be even.
- *Hence*: ***q must be odd.*** ← *If q were even, then since p is even, p/q would not be in lowest terms!*
- *So*: $p = 2r$, for some integer r.
- *So*: $2q^2 = (2r)^2 = 4r^2$.
- *Or*: $q^2 = 2r^2$
- *So*: q^2 is even.
- *But*: This means ***q must be even!***

3. The Eleatics

Parmenides of Elea (515 B.C.)

Claim: It is meaningless to speak of what is not. Everything *is*.

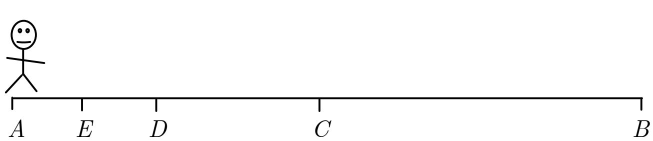
→ "***The One***" - the metaphysically infinite
- indivisible, homogeneous, eternal

Further claim: Change is an illusion. (Change is a transition from what is, to what is not. This is impossible, since talk of what is not is incoherent.)

Zeno (490 B.C.)

Paradoxes of motion: ← *intended to demonstrate that motion is not real*

- Achilles and the Tortoise
- ***Paradox of the runner:***



Claim: Achilles will never reach the finish-line at B.

Proof:
(1) To reach B, must reach $C = AB/2$.
(2) To reach C, must reach $D = AC/2$, *etc...*
(3) Thus there are an infinite number of finite line segments between A and B.
(4) So Achilles would need an infinite amount of time to traverse them all!

Assumptions:
(a) AB is infinitely divisible.
(b) The sum of an infinite number of finite lengths is infinite.

4. Plato (428 - 347 B. C.)

"***a peiron***" - indeterminacy/disorder/"chaos"

"***peras***" - limits/order

3 places where notions of the infinite appear in Plato:

1. Account of creation of physical world: Result of imposing Forms on *indeterminacy* to produce order
2. *Eternal* nature of the World of Forms.
3. *Infinite diversity* in the Physical World.

II. The Method of Exhaustion (Eudoxus and Archimedes)

Euclidean Geometry: Two notions of the infinite:

(a) infinite divisibility of line segments

(b) infinite extendability of line segments

lead to paradoxes of infinitely small and infinitely big

BUT: Early Greeks tended to avoid talk of the infinite. In geometry, all objects are really finite (like natural numbers: any one is finite; together are all infinite).

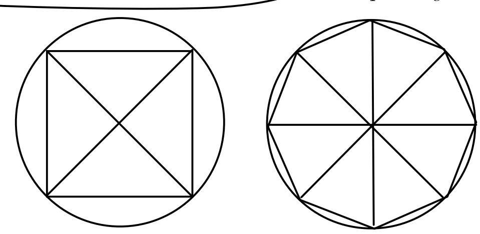
Example: ***Method of Exhaustion*** (Eudoxus 408-355 BC) as used by Archimedes to prove area of circle = πr^2.

Let C be a circle with radius r.

For each natural number n, let P_n be a regular polygon inscribed in C.

n equal sides and n equal angles

Divide P_n into n congruent triangles.



P_4 P_8

Let b_n = base of triangle

h_n = height of triangle

THEN: area of triangle = $1/2\, b_n h_n$

AND: area of $P_n = 1/2\, n b_n h_n$

Now visualize C as P_∞ -- a polygon with *infinitely many infinitely small* sides.

SO: When $n = \infty$:

$$nb_n = (\text{circumference of } C) = 2\pi r$$

$$h_n = r$$

The height of each (infinitely thin!) triangle in P_∞ is identified with the radius of C (and the base of each triangle in P_∞ is very, very small... infinitely small!).

SO: area of C = area of P_∞

$= 1/2\, n b_n h_n$, when $n = \infty$

$= 1/2(2\pi r)r$

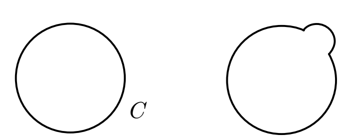
$= \pi r^2$

Problems

(1) What does it mean to multiply by an infinitely small amount (b_n when $n = \infty$)? (Can't be same as multiplying by 0!)

(2) What is a polygon with *infinitely many infinitely small sides*?

(3) As n goes to infinity, P_n approximates C, but also C^*:



C C^*

What does it mean to say C is what P_n is tending towards and not C?*

Archimedes' Solution

Proved 2 claims:

Claim I: There is a regular polygon as close in area to C as you care to specify. ($i.e.$, For any arbitrary small area ε, there is always a number n such that P_n differs in area from C by less than ε.)

not true for C^*

Consequence: The area of C is at most πr^2.

Claim II: The area of C is at least πr^2.

Consequence of I and II: The area of C is exactly πr^2.

Significance of Archimedes' Solution: No mention of infinity!

III. Aristotle

Empiricist: Platonic Forms are in the physical world. (*only world there is for Aristotle*)

Relevant Question: Is anything in nature ***infinite***? ("*untraversible*")

Aristotle's Answer: ***The infinite exists potentially and not actually.***

actual infinite: that whose infinitude exists at some point in *time*

potential infinite: that whose infinitude exists *over* time (not wholly present)

Note: For A., this is *literally* the distinction: Time is infinite, but not space.

Aristotle's Response to Zeno's Paradoxes:

Achilles and the Tortoise

The distance between Achilles and the Tortoise is only *potentially* infinitely divisible; it is not *actually* infinitely divisible. And there is no contradiction in claiming that a finite length is potentially infinitely divisible.

or:

To travel a *potentially* infinitely divisible distance, Achilles needs a *potentially* infinite time. And there is nothing wrong in claiming he has such a time available.

$$\begin{pmatrix}\text{potentially infinitely} \\ \text{divisible length } \ell\end{pmatrix} : \begin{pmatrix}\text{for any } n\text{, it is } \textit{possible} \text{ to divide} \\ \ell \text{ into more than } n \text{ parts}\end{pmatrix}$$

$$\begin{pmatrix}\text{actually infinitely} \\ \text{divisible length } \ell\end{pmatrix} : \begin{pmatrix}\ell \ \textit{can} \text{ be divided into infinitely many parts}\end{pmatrix}$$

Aristotle's infinite: "the untraversible"

"Something is infinite if, taking it quantity by quantity, we can always take something outside."

"It is not what has no part outside it that is infinite, but what always has some part outside it."

Under Moore's reading, Aristotle rejects the "metaphysically infinite" and adopts the "mathematically infinite".

Problem for Aristotle: What about the infinite past?

Already traversed?
So actually infinite?

Wittgenstein's Story:

Suppose we come across a man saying "... 5, 1, 4, 3.", who then proceeds to tell us that he has just finished reciting π backwards for all past eternity. Why does this strike us as impossible, whereas someone who just starts reciting π forwards and will continue for all future eternity does not (given that we concede the possibility of living forever).