02. Paradoxes

The Infinite -- Two clusters of concepts:

- boundlessness
- endlessness
- unlimitedness
- immeasurability
- eternity
- that which is greater than any assignable quantity

- completeness
- wholeness
- unity
- universality
- absoluteness
- perfection
- self-sufficiency
- autonomy

- *"negative" characteristics*
- *potentiality*
- **mathematical**

- *"positive" characteristics*
- *actuality*
- **metaphysical/theological**

Paradoxes
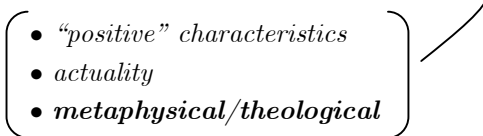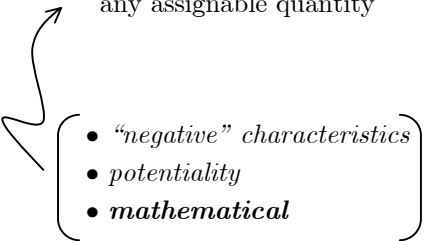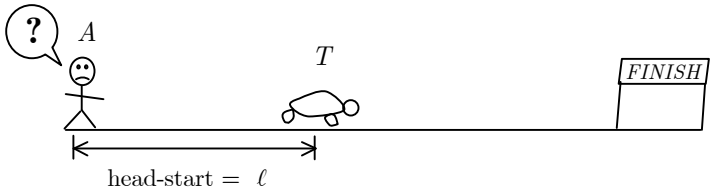
1. Paradoxes of the infinitely small

2. Paradoxes of the infinitely big

3. Paradoxes of the one and the many

4. Paradoxes of thought about the infinite

I. Paradoxes of the Infinitely Small

Ex 1. Achilles and the Tortoise

A runs at speed $v_A = \ell/t$

T runs at speed $v_T = v_A/2 = (\ell/2)/t$

head-start $= \ell$

Claim: Achilles will never overtake the tortoise.

Proof: (1) To overtake T, A must first travel ℓ, which takes him time t.
   (2) In time t, T travels $\ell/2$.
   (3) To travel $\ell/2$, A needs further time $t/2$.
   (4) In time $t/2$, T travels $\ell/4$.
   (5) To travel $\ell/4$, A needs even more time, $t/4$. *Etc...*

In general:

The distance between A and T at any given moment after the start of the race is finite (even though it's approaching 0).

And:

To travel a finite distance at finite speed requires a finite amount of time.

important Euclidean assumption : [*A line segment is infinitely divisible*]
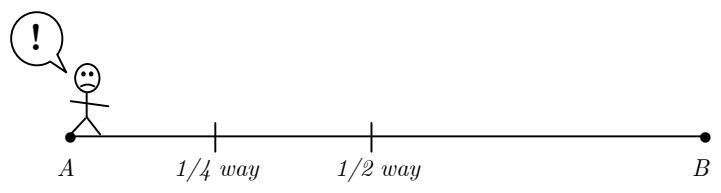
Ex 2. The Staccato Run

Achilles runs for 1/2 minute,
then rests for 1/2 minute,
then runs for 1/4 minute,
then rests for 1/4 minute,
etc...

Claim: After 2 minutes, Achilles will have stopped and started an *infinite* number of times!
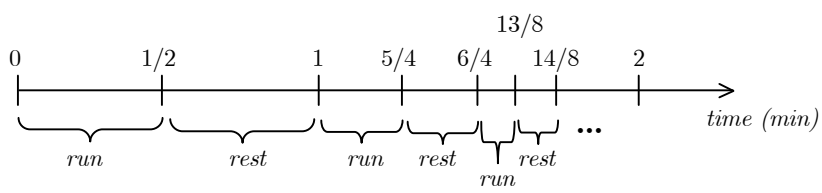
Model a period of time (2 minutes) as a line segment, which is infinitely divisible (according to Euclid).

Ex 3. The Paradox of the Gods

Achilles desires to run from A to B.
God #1's intent: To paralyze Achilles when he gets 1/2 way.
God #2's intent: To paralyze Achilles when he gets 1/4 way.
God #3's intent: To paralyze Achilles when he gets 1/8 way.
Etc...

infinite # of gods

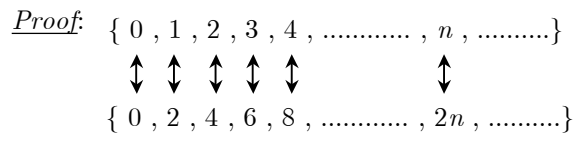*Claim*: Achilles will be unable to move!

But Why?: Until he moves, none of the (infinite) gods will have actually paralyzed him!

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II. Paradoxes of the Infinitely Big

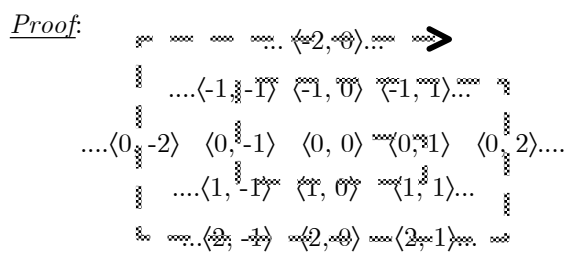### Ex. 1. The Paradox of the Even Numbers

Claim: There are as many even natural numbers as there are natural numbers.

> *Aside*: Two sets have the same number of members just when there is a 1-1 correspondence between their members.

Proof:
$$\{\, 0 \,,\, 1 \,,\, 2 \,,\, 3 \,,\, 4 \,,\, ... \,,\, n \,,\, ...\}$$
$$\updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \updownarrow\ \ \updownarrow \qquad\qquad \updownarrow$$
$$\{\, 0 \,,\, 2 \,,\, 4 \,,\, 6 \,,\, 8 \,,\, ... \,,\, 2n \,,\, ...\}$$

Ex. 2. The Paradox of the Pairs

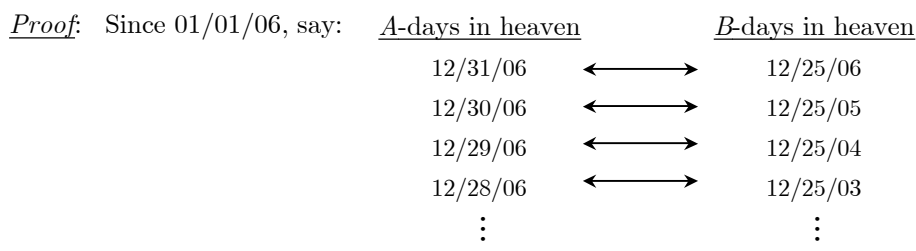*Claim*: There are as many pairs of whole numbers as there are natural numbers.

Consequence: There are as many rational numbers as natural numbers. *(Recall that a rational number is given by a pair of whole numbers).*

Proof:

Path assigns every pair to one natural number

Ex. 3. The Paradox of Two Guys in Heaven and Hell

For all past eternity:

Mr. A in heaven
Mr. B in hell
$\Big\}$ except for one day each year when they switch (Christmas Day, say)

Claim: Mr. B has spent just as much time in heaven as Mr. A.

Proof: Since 01/01/06, say:

A-days in heaven		B-days in heaven
12/31/06	⟷	12/25/06
12/30/06	⟷	12/25/05
12/29/06	⟷	12/25/04
12/28/06	⟷	12/25/03
⋮		⋮

Ex 4. Hilbert's Hotel

Consider a hotel with an infinite number of rooms, all occupied.

   Claim 1: One more newcomer can be accomodated.

     Proof: Require all current occupants to move to next room.

   Claim 2: Infinitely more newcomers can be accomodated!

     Proof: Require current occupants to move accordingly:

> 1st occupant moves into 2nd room.
> 2nd moves into 4th.
> 3rd moves into 6th.
> 4th moves into 8th.
> 5th moves into 10th.
> 6th moves into 12th.
> *etc...*

opens up all infinitely many odd-numbered rooms!

III. Paradoxes of the One and the Many

   In general: Can a collection of infinitely many things be considered a single thing?

  Set Theory: What exactly is a set?

      Cantor: "a many which allows itself to be thought of as a one"

  Are there infinite sets?   We will return to
this question later.

IV. Paradoxes of Thought About the Infinite

  Moore: In general, is the concept of infinity coherent? Yes and No.

Yes	*No*
• *want to be able to say there are infinitely many natural numbers*	• *prior paradoxes*
• *want to be able to say the world "includes" everything (infinitely inclusive)*	

   Moore's suggestion: Admit concept of infinite, but
acknowledge that we as finite beings
cannot come to know it.

How can we grasp
the ungraspable?

But we do know it!