Homework #1. Due: Thurs 9/14

1. Calculate the following products (you may need to consult the lecture notes online):
   (a) *Real number multiplication.* 3×4
   (b) *Complex number multiplication.* $(3 + i2) \times (4 + i)$
   (c) *Quaternion multiplication.* $(3 + i2 + j + k2) \times (4 + i + j3 + k)$
   (d) *Octonion multiplication.* $(2 + e_5 3) \times (1 + e_7 4)$

2. Is $2 + e_5 3$ an octonion? Is it a quaternion? Is it a complex number? Explain your responses. (*Hint*: To answer these questions, keep in mind the definitions of octonion, quaternion, and complex number. In particular, what are the *general forms* for each of these numbers, and does the number above fit these general forms?)

3. Here are some consequences of the multiplication rule, when applied to positive and negative numbers:
   (i) a *positive number* times a *positive number* is a *positive number*;
   (ii) a *negative number* times a *positive number* is a *negative number*;
   (iii) a *negative number* times a *negative number* is a *positive number*.

   Here's the motivation for (i): Let m, n be positive numbers. Then $m \times n$ is what you get when you add m to itself n-times. For instance, $2 \times 3 = \underbrace{2 + 2 + 2}_{\text{3-times!}} = 6$.

   This also motivates (ii):
   $-2 \times 3 = \underbrace{(-2) + (-2) + (-2)}_{\text{3-times!}} = -6$

   What about (iii)? Note first that $-m \times -n$ can be written as $(-1 \times m) \times (-1 \times n) = (-1 \times -1) \times (m \times n)$. So (iii) boils down to the question of what to write for -1×-1. Show that this has to be $+1$, by rewriting -1 as the difference between two positive numbers. (In other words, rewrite $-1 = p - q$, for an appropriate choice of positive numbers p and q. Then substitute this difference into -1×-1.)

4. In Hilbert's Hotel, suppose all the infinite rooms are occupied, each with one bar of soap. Now suppose each guest wants an infinite number of bars of soap. Explain in as much detail as possible how Hilbert can shuttle people and existing soap bars around so that each person ends up with an infinite number of soap bars. (*Hint*: Initially there are an infinite number of people, each with a bar of soap. How can you move the people around so that each person still has a bar of soap, but there are now an infinite number of unoccupied rooms each with a single bar of soap? What else do you now have to do?)