Extra credit#1. Due: Thurs 11/30

pg. 54, Exercise #8

Big Hint: You are given two maps that can be composed, and sections of these maps. Call the two maps f and g. That they can be composed means that it's possible to first do one, say f, and then the other, say g. This means that the codomain of f must be the domain of g. So if we let $f: A \to B$, then $g : B \to C$, for instance. Then the composite map $g \circ f: A \to C$ is well-defined. To say that both f and g have sections is to say that there are two other maps, call them s_1, s_2, such that $f \circ s_1 = 1_B$ and $g \circ s_2 = 1_C$. All of this means that we are given the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left=50, "g \circ f"] & B \arrow[l, bend left, "s_1"] \arrow[r, "g"] & C \arrow[l, bend left, "s_2"] \arrow[ll, dashed, bend left=50, "s = ?"] \end{tikzcd} \qquad \left.\begin{array}{l} f \circ s_1 = 1_B \\ g \circ s_2 = 1_C \end{array}\right\} \ \underline{\textit{given}}$$

Now we are asked to show that the composite map $g \circ f$ has a section, call it s. So you first need to come up with an appropriate expression for s (read it off of the diagram: s is supposed to go from C to A; what let's you do this?); and then you have to show that this expression for s satisfies the definition of being a section for $g \circ f$. Namely, that $(g \circ f) \circ s = 1_C$.

pg. 54, Exercise #9

Big Hint: Part I. Let $f: A \to B$. If r is a retraction for f, then it satisfies $r \circ f = 1_A$. Use this to show that $e = f \circ r$ is an *idempotent*. So you have to show that $e \circ e = e$. The outline of your proof should look like:

$$\begin{aligned} e \circ e &= (f \circ r) \circ (f \circ r) && \textit{given} \\ &= \quad . && . \\ & \quad\ \ . && . \\ & \quad\ \ . && . \\ &= e \end{aligned}$$

For Part II, you have to show that, if f is an isomorphism, then e is the identity. We know that e goes from B to B (why?). So you need to show that $e = 1_B$. If f is an isomorphism, then it has an inverse, call it f^{-1}, that satisfies $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. The outline of your proof should look like:

$$\begin{aligned} e &= e \circ 1_B && \textit{identity} \\ &= \quad . && . \\ & \quad\ \ . && . \\ & \quad\ \ . && . \\ &= 1_B \end{aligned}$$

pg. 55, Exercise #10

Small Hint: Since f and g are isomorphisms, they have inverses, call them f^{-1}, g^{-1}. They satisfy $f \circ f^{-1} = 1_B$, $f^{-1} \circ f = 1_A$, $g \circ g^{-1} = 1_C$, $g^{-1} \circ g = 1_B$. Now you need to show that $g \circ f$ has an inverse, call it $(g \circ f)^{-1}$, and that it is given by $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. So you need to construct two proofs, one demonstrating that this expression for $(g \circ f)^{-1}$ satisfies $(g \circ f) \circ (g \circ f)^{-1} = 1_C$, and the other demonstrating that this expression for $(g \circ f)^{-1}$ satisfies $(g \circ f)^{-1} \circ (g \circ f) = 1_A$ (you should check that the identities here are appropriate). See if you can construct a diagram that represents all the data that you're given in this problem (it'll look a little like the diagram for Exercise #8 above).