Knowledge assumed in this document:

Algebraic definition of even, odd, irrational.

IF-THEN, IFF, $\rightarrow$, $\leftrightarrow$

$\exists$

$\mathbb{Z}$

prime & prime factor

every integer is a product of primes (see document on induction)

if Greece wins the world cup, I will be happy (forever) if Greece wins the world cup, I will be happy (forever)

if I'm not happy, Greece has not won the world cup

CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

equivalent

if I'm not happy, Greece has not won the world cup

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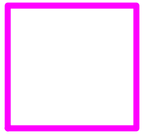
if you are a square, you have corners 

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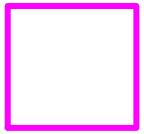
if you are a square, you have corners 

equivalent

?

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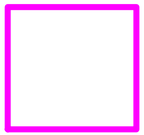
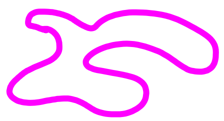
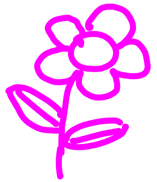
if you are a square, you have corners 

equivalent

if you don't have corners, you are not a square  

CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

$\Updownarrow$ equivalent $\Updownarrow$

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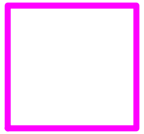
if you are a square, you have corners 

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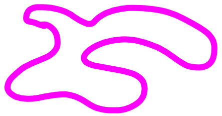
if you don't have corners, you are not a square  

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if A then B $\Longleftrightarrow$?

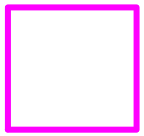
CONTRAPOSITIVE

if Greece wins the world cup, I will be happy (forever)

↕ equivalent ↕

if I'm not happy, Greece has not won the world cup

---

if you are a square, you have corners 

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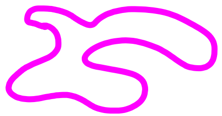
if you don't have corners, you are not a square  

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if A then B $\Longleftrightarrow$ if not B, then not A

CONTRAPOSITIVE: if A then B = if not B, then not A

CONTRAPOSITIVE:

if A then B = if not B, then not A

$A \to B = \neg B \to \neg A$

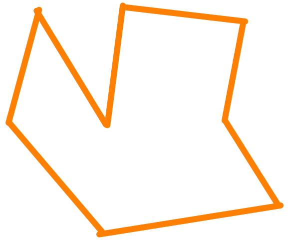
not B → not A

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \quad = \quad \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

Greece hasn't won, but I'm still happy



This shape isn't a square, but it has corners

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \quad = \quad \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

CONTRAPOSITIVE: if A then B = if not B, then not A

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a	b
T	T
T	F
F	T
F	F

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \quad = \quad \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

a	b	$a \to b$ valid?
T	T	?
T	F	
F	T	
F	F	

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B = \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

a	b	$a \to b$ valid?
T	T	✓
T	F	?
F	T	
F	F	

CONTRAPOSITIVE: if A then B = if not B, then not A

$A \rightarrow B = \neg B \rightarrow \neg A$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

a	b	$a \rightarrow b$ valid?
T	T	✓
T	F	✗
F	T	?
F	F	?

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \rightarrow B = \neg B \rightarrow \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

a	b	$a \rightarrow b$ valid?
T	T	✓
T	F	✗
F	T	✓
F	F	✓

don't contradict

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \rightarrow B = \neg B \rightarrow \neg A$$

What if $\neg A$ holds, but B is still true?

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a	b	$a \rightarrow b$ valid?	$\neg b$	$\neg a$
T	T	✓	F	F
T	F	✗	T	F
F	T	✓	F	T
F	F	✓	T	T

don't contradict

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B = \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's ok; no contradiction. It's not B IFF A

	a	b	$a \to b$ valid?	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$ valid?
	T	T	✓	F	F	
	T	F	✗	T	F	
don't contradict	F	T	✓	F	T	
	F	F	✓	T	T	?

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \quad = \quad \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

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a	b	$a \to b$ valid?	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$ valid?
T	T	✓	F	F	
T	F	✗	T	F	?
F	T	✓	F	T	
F	F	✓	T	T	✓

don't contradict

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B = \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

↳ That's OK; no contradiction. It's not B IFF A

	a	b	a→b valid?	¬b	¬a	(¬b)→(¬a) valid?
	T	T	✓	F	F	?
	T	F	✗	T	F	✗
don't contradict	F	T	✓	F	T	?
	F	F	✓	T	T	✓

CONTRAPOSITIVE: if A then B = if not B, then not A

$$A \to B \quad = \quad \neg B \to \neg A$$

What if $\neg A$ holds, but B is still true?

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a	b	$a \to b$ valid?	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$ valid?
T	T	✓	F	F	✓
T	F	✗	T	F	✗
F	T	✓	F	T	✓
F	F	✓	T	T	✓

don't contradict

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a	b	$a \to b$ valid?	$\neg b$	$\neg a$	$(\neg b) \to (\neg a)$ valid?
T	T	✓	F	F	✓
T	F	✗	T	F	✗
F	T	✓	F	T	✓
F	F	✓	T	T	✓

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

$\neg$ corners $\to$ $\neg$ square

Context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Context so far: we know $A \to B$, so if we observe $\neg B$
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Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

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direct

$7x+9 = 2a$ // a: integer $\to$ $7x+9$: even

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

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$7x+9 = 2a$ // a: integer $\to$ $7x+9$: even

$x = 2a - 6x - 9$

context so far: we know $A \to B$, so if we observe $\neg B$
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direct

$7x+9 = 2a$ // a: integer $\to$ $7x+9$: even

$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

context so far: we know $A \rightarrow B$, so if we observe $\neg B$
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direct

$7x+9 = 2a$ // a: integer → 7x+9 : even

$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

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$x = 2b+1$ $(b = a - 3x - 5)$

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

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$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

$x = 2b + 1$ (odd) ($b = a - 3x - 5$) $\square$

end of proof.

context so far: we know $A \to B$, so if we observe $\neg B$
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PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct	contrapositive
$7x+9 = 2a$ // a: integer → 7x+9 : even	
$x = 2a - 6x - 9$	
$x = 2a - 6x - 10 + 1$	
$x = 2(a - 3x - 5) + 1$	
$x = 2b+1$ (odd) $(b = a-3x-5)$ $\square$	

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

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direct

$7x+9 = 2a$ // a: integer $\to$ $7x+9$: even

$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

$x = 2b + 1$ (odd) $(b = a - 3x - 5)$ $\square$

contrapositive

Suppose ... ?

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct	contrapositive
$7x+9 = 2a$ // a: integer → 7x+9 : even	Suppose x is not odd: $x = ?$
$x = 2a - 6x - 9$	
$x = 2a - 6x - 10 + 1$	
$x = 2(a - 3x - 5) + 1$	
$x = 2b+1$ (odd) $(b = a - 3x - 5)$ $\square$	

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

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We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct

$$7x+9 = 2a \quad // a\text{: integer} \to 7x+9\text{: even}$$
$$x = 2a - 6x - 9$$
$$x = 2a - 6x - 10 + 1$$
$$x = 2(a - 3x - 5) + 1$$
$$x = 2b+1 \text{ (odd) } (b = a - 3x - 5) \quad \square$$

contrapositive

Suppose x is not odd: $x = 2c$

... then?

context so far: we know $A \to B$, so if we observe $\neg B$
then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct

$7x+9 = 2a$ // a: integer $\to$ $7x+9$: even

$x = 2a - 6x - 9$

$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

$x = 2b+1$ (odd) $(b = a - 3x - 5)$ $\square$

contrapositive

Suppose x is not odd: $x = 2c$

$7x + 9 = 7 \cdot 2c + 9$

Context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

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$x = 2(a - 3x - 5) + 1$

$x = 2b+1$ (odd) $(b = a - 3x - 5)$ $\square$

contrapositive

Suppose x is not odd: $x = 2c$

$7x + 9 = 7 \cdot 2c + 9$

$= 14c + 8 + 1$

context so far: we know $A \to B$, so if we observe $\neg B$
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$x = 2(a - 3x - 5) + 1$

$x = 2b+1$ (odd) $(b = a - 3x - 5)$ $\square$

contrapositive

Suppose x is not odd: $x = 2c$

$$\begin{aligned}7x+9 &= 7 \cdot 2c + 9 \\ &= 14c + 8 + 1 \\ &= 2 \cdot (7c+4) + 1\end{aligned}$$

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $7x+9$ is even, then x is odd (for $x \in \mathbb{Z}$)

direct

$7x+9 = 2a$ // a: integer $\to 7x+9$: even

$x = 2a-6x-9$

$x = 2a-6x-10+1$

$x = 2(a-3x-5)+1$

$x = 2b+1$ (odd) $(b=a-3x-5)$ $\square$

contrapositive

Suppose x is not odd: $x = 2c$

$7x+9 = 7\cdot 2c+9$

$= 14c+8+1$

$= 2\cdot(7c+4)+1$

$= 2\cdot d+1$ $(d=7c+4)$

context so far: we know $A \to B$, so if we observe $\neg B$ then we can conclude $\neg A$

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$x = 2a - 6x - 10 + 1$

$x = 2(a - 3x - 5) + 1$

$x = 2b+1$ (odd) $(b = a-3x-5)$ $\square$

contrapositive

Suppose x is not odd: $x = 2c$

$7x+9 = 7 \cdot 2c + 9$

$= 14c + 8 + 1$

$= 2 \cdot (7c+4) + 1$

$= 2 \cdot d + 1$ $(d = 7c+4)$

$7x+9 = \text{odd}$ $\square$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

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phrase mathematically?

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We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

phrase mathematically?

$$(x^2 - 6x + 5 = 2a) \longrightarrow (x = 2b + 1)$$

& x, a, b are integers

PROOF BY CONTRAPOSITIVE

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Prove: if $x^2 - 6x + 5$ is even, then x is odd

$x^2 - 6x + 5 = 2a$ direct

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

$x^2 - 6x + 5 = 2a$ direct

$\vdots$?

$\downarrow$

$x = 2b + 1$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

$$x^2 - 6x + 5 = 2a$$

direct

$$x^2 - 6x + (5 - 2a) = 0$$

$$x = \frac{6 \pm \sqrt{36 + 8a - 20}}{2}$$

$$x = 3 \pm \sqrt{4 + 2a}$$

$\vdots$?

$$x = 2b + 1$$

PROOF BY CONTRAPOSITIVE

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Prove: if $x^2 - 6x + 5$ is even, then x is odd

direct	contrapositive
$x^2 - 6x + 5 = 2a$	...?
$\vdots$?	
$x = 2b + 1$	

PROOF BY CONTRAPOSITIVE

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direct	contrapositive
$x^2 - 6x + 5 = 2a$	Suppose x is not odd: $x = 2c$
$\vdots$?	
$x = 2b + 1$	

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

direct	contrapositive
$x^2 - 6x + 5 = 2a$	Suppose x is not odd: $x = 2c$
$\vdots$?	$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$
$x = 2b + 1$	

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

direct

$x^2 - 6x + 5 = 2a$

$\vdots$?

$x = 2b + 1$

contrapositive

Suppose x is not odd: $x = 2c$

$$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$$
$$= 4c^2 - 12c + 5$$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if $x^2 - 6x + 5$ is even, then x is odd

direct	contrapositive
$x^2 - 6x + 5 = 2a$	Suppose x is not odd: $x = 2c$
$\vdots$?	$x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$
$x = 2b + 1$	$= 4c^2 - 12c + 5$
	$= 4c^2 - 12c + 4 \underline{+1}$
	because we want to get something odd

PROOF BY CONTRAPOSITIVE

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direct	contrapositive
$x^2 - 6x + 5 = 2a$ $\vdots$? $x = 2b + 1$	Suppose x is not odd: $x = 2c$ $x^2 - 6x + 5 = (2c)^2 - 6 \cdot 2c + 5$ $= 4c^2 - 12c + 5$ $= 4c^2 - 12c + 4 + 1$ $= 2 \cdot (2c^2 - 6c + 2) + 1$

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$x = 2b + 1$	$= 2 \cdot d + 1 \quad (d = 2c^2 - 6c + 2)$

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	$= 2 \cdot d + 1 \quad (d = 2c^2 - 6c + 2)$
$x = 2b + 1$	$=$ not even $\square$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove: if x is irrational then $\sqrt{x}$ is irrational

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We don't know how to prove $A \to B$ (easily), so we try to start by <u>assuming $\neg B$</u>. If we <u>conclude $\neg A$</u>, we are done.

Prove: if x is irrational then $\sqrt{x}$ is irrational

direct

???

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???	Suppose $\sqrt{x}$ is not irrational $\sqrt{x} = \frac{a}{b}$ $\quad a, b \in \mathbb{Z}$

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direct	contrapositive
???	Suppose $\sqrt{x}$ is not irrational $\sqrt{x} = \frac{a}{b}$ $\quad a,b \in \mathbb{Z}$ $x = \frac{a^2}{b^2}$

PROOF BY CONTRAPOSITIVE

We don't know how to prove $A \to B$ (easily), so we try to start by assuming $\neg B$. If we conclude $\neg A$, we are done.

Prove : if x is irrational then $\sqrt{x}$ is irrational

direct	contrapositive
???	Suppose $\sqrt{x}$ is not irrational
	$\sqrt{x} = \frac{a}{b}$ $\quad a,b \in \mathbb{Z}$
	$x = \frac{a^2}{b^2}$: not irrational $\square$

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

---

still proving if A then B for now

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

---

still proving **if A then B** for now

$$(a+b)\cdot(a-b) \longrightarrow a^2-ab+ba-b^2 \longrightarrow a^2-b^2$$

You can prove something directly (in one direction)

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

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still proving **if A then B** for now

$$(a+b)\cdot(a-b) \longrightarrow a^2-ab+ba-b^2 \longleftarrow a^2-b^2$$

You can prove something directly (in one direction)
or work in both directions

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

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still proving if A then B for now

$(a+b)\cdot(a-b) \longleftrightarrow a^2-ab+ba-b^2 \longleftrightarrow a^2-b^2$

You can prove something directly (in one direction)
or work in both directions

contrapositive $\Big\{$ starting w/ $\neg B$ & leading to $\neg A$
contradicts $A \to \neg B$

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving **if A then B** for now

$$(a+b)\cdot(a-b) \longleftrightarrow a^2-ab+ba-b^2 \longleftrightarrow a^2-b^2$$

You can prove something directly (in one direction) or work in both directions

instead of starting w/ $\neg B$ & leading to $\neg A$
(which contradicts $A \to \neg B$)

PROOF BY CONTRADICTION

a slight generalization of proof by contrapositive

still proving if A then B for now

$$(a+b)\cdot(a-b) \longleftrightarrow a^2 - ab + ba - b^2 \longleftrightarrow a^2 - b^2$$

You can prove something directly (in one direction)
or work in both directions

instead of starting w/ $\neg B$ & leading to $\neg A$
(which contradicts $A \to \neg B$)

$$\boxed{\text{assume both } A \text{ and } \neg B \text{ are true \& arrive at some contradicting statement}}$$

PROOF BY CONTRADICTION

If x is even then x is not odd

PROOF BY CONTRADICTION

If $\underbrace{\text{x is even}}_{A}$ then $\underbrace{\text{x is not odd}}_{B}$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \underbrace{\wedge}_{\text{and}} \neg B$: ...

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: $\quad x$ is even $\quad$ & $\quad x$ is odd

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$? \qquad\qquad\qquad ?$$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

$\downarrow$ $\downarrow$

(a: int.) $x = 2a$ $x = 2b+1$ (b : int.)

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: $\quad x$ is even $\quad$ & $\quad x$ is odd

$$x \text{ is even} \downarrow x = 2a \quad (a: \text{int.})$$

$$x \text{ is odd} \downarrow x = 2b+1 \quad (b: \text{int.})$$

$$2a = 2b+1$$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

(a: int.) $x = 2a$ $\qquad$ $x = 2b+1$ (b: int.)

$$2a = 2b+1$$

$$a = b + \frac{1}{2}$$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

$$x \text{ is even} \Rightarrow x = 2a \quad (a: \text{int.})$$

$$x \text{ is odd} \Rightarrow x = 2b+1 \quad (b: \text{int.})$$

$$2a = 2b+1$$

$$a = b + \tfrac{1}{2}$$

impossible/absurd/contradiction $\square$

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

$\downarrow$ $\downarrow$

(a: int.) $x = 2a$ $x = 2b+1$ (b: int.)

$$2a = 2b+1$$

$$a = b + \frac{1}{2}$$

impossible/absurd/contradiction □

Notice we met halfway at an incorrect statement.

PROOF BY CONTRADICTION

If $\underbrace{x \text{ is even}}_{A}$ then $\underbrace{x \text{ is not odd}}_{B}$

Assume $A \wedge \neg B$: x is even & x is odd

$$x \text{ is even} \downarrow \quad x = 2a \quad (a\text{: int.})$$

$$x \text{ is odd} \downarrow \quad x = 2b+1 \quad (b\text{: int.})$$

$$2a = 2b+1$$

$$a = b + \tfrac{1}{2}$$

impossible / absurd / contradiction $\square$

Could also plug $b+\frac{1}{2}$ into $x=2a$ & conclude x is odd.

Notice we met halfway at an incorrect statement.

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number x s.t. $ax + b = 0$.

state this in IF-THEN form

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $ax+b=0$.

(if $ax+b=0$ then for $y \neq x$, $ay+b \neq 0$)

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $ax+b=0$.

(if $\underbrace{ax+b=0}_{A}$ then for $y \neq x$, $\underbrace{ay+b \neq 0}_{B}$)

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $ax+b=0$.

$$\left(\text{if } \underbrace{ax+b=0}_{A} \text{ then for } y \neq x,\ \underbrace{ay+b \neq 0}_{B} \right)$$

Assume $A \wedge \neg B$: $\quad ax+b=0 \quad \& \quad ay+b=0$

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $ax + b = 0$.

(if $\underbrace{ax+b=0}_{A}$ then for $y \neq x$, $\underbrace{ay+b \neq 0}_{B}$)

Assume $A \wedge \neg B$: $\quad ax+b=0 \quad \& \quad ay+b=0$

$$ax+b = ay+b$$

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $ax + b = 0$.

$$\left(\text{if } \underbrace{ax+b=0}_{A} \text{ then for } y \neq x,\ \underbrace{ay+b \neq 0}_{B}\right)$$

Assume $A \wedge \neg B$: $ax+b=0$ & $ay+b=0$

$$ax+b = ay+b$$

$$ax = ay$$

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $ax + b = 0$.

$$\left(\text{if } \underbrace{ax+b=0}_{A} \text{ then for } y \neq x,\ \underbrace{ay+b \neq 0}_{B}\right)$$

Assume $A \wedge \neg B$:

$$ax+b=0 \quad \& \quad ay+b=0$$

$$ax+b = ay+b$$

$$ax = ay$$

$$x = y$$

PROOF BY CONTRADICTION

For integers $a \neq 0$ & b, there is only one number s.t. $az + b = 0$.

$$\left(\text{if } \underbrace{ax+b=0}_{A} \text{ then for } y \neq x,\ \underbrace{ay+b \neq 0}_{B}\right)$$

Assume $A \wedge \neg B$: $\quad ax+b=0 \quad \& \quad ay+b=0$

$$ax+b = ay+b$$

$$ax = ay$$

$$x = y$$ — contradicts $\square$

Prove: if A then B

Assume $A \wedge \neg B$, get contradiction. ✓

Prove: if A then B

Assume $A \wedge \neg B$, get contradiction. ✓

Does it work if we assume $\neg A \wedge B$ and get a contradiction?

Prove: if A then B

Assume $A \wedge \neg B$, get contradiction. ✓

Does it work if we assume $\neg A \wedge B$ and get a contradiction?

NO

$A \to B$ tells us nothing about what happens when $\neg A$.

Prove: if A then B

Assume $A \wedge \neg B$, get contradiction. ✓

Does it work if we assume $\neg A \wedge B$ and get a contradiction?

NO

$A \rightarrow B$ tells us nothing about what happens when $\neg A$.

It would work if we were proving $A \leftrightarrow B$

Let's prove something not in IF-THEN format

$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

1) $\nexists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

1) $\nexists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

$\Updownarrow$ $\sqrt{2}$ is rational

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

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1) $\nexists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a, b\}$ w/ no common divisor

(simplify)

$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ <u>no common divisor</u>

By (2), $2 = \frac{a^2}{b^2}$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

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3) if (2) is true, then choose $\{a,b\}$ w/ <u>no common divisor</u>

By (2), $2 = \frac{a^2}{b^2} \Rightarrow \boxed{2b^2 = a^2}$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

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2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ <u>no common divisor</u>

By (2), $2 = \frac{a^2}{b^2} \Rightarrow \underbrace{2}b^2 = a^2 \Rightarrow a^2$: $\underbrace{\text{even}}$

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1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow$ a^2: even (a: even)

why?

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

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3) use this to establish something that you know is wrong

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3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow \boxed{\begin{array}{l} a^2 : \text{even} \\ (a : \text{even}) \end{array}}$

$$(2x+1)\cdot(2x+1) = 4x^2+4x+1$$
$$= 2\cdot(2x^2+2x)+1$$

$a:\text{odd} \rightarrow a^2:\text{odd}$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) $\nexists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a, b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a, b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even
(a: even)

$a = 2c$ $\{c: \text{int}\}$

$\sqrt{2}$ IS IRRATIONAL - PROOF BY CONTRADICTION

1) what does the claim mean?

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By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even (a: even)

$a = 2c$ $\{c: \text{int}\}$ $\Rightarrow 2b^2 = 4c^2$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

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By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even
(a: even)

$a = 2c$ $\{c: \text{int}\}$ $\Rightarrow 2b^2 = 4c^2 \Rightarrow b$: even

$(b^2 = 2c^2 \Rightarrow b^2$: even$)$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

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3) use this to establish something that you know is wrong

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3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even (a: even)

$a = 2c \; \{c : \text{int}\} \Rightarrow 2b^2 = 4c^2 \Rightarrow b$: even ($b = 2d$)

$\sqrt{2} = \frac{a}{b} = \frac{2c}{2d}$

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

1) $\not\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even
(a: even)

$a = 2c$ $\{c : \text{int.}\} \Rightarrow 2b^2 = 4c^2 \Rightarrow b$: even

$\sqrt{2} = \frac{a}{b} = \frac{2c}{2d}$ contradiction

$\sqrt{2}$ IS IRRATIONAL – PROOF BY CONTRADICTION

1) what does the claim mean?

2) assume the contrary is true

3) use this to establish something that you know is wrong

4) conclude that (2) is false thus the initial claim is true

1) $\nexists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

2) $\exists$ integers $\{a,b\}$ s.t. $\sqrt{2} = \frac{a}{b}$

3) if (2) is true, then choose $\{a,b\}$ w/ no common divisor

By (2), $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow a^2$: even (a: even)

$a = 2c \; \{c: \text{int}\} \Rightarrow 2b^2 = 4c^2 \Rightarrow b$: even

$\sqrt{2} = \frac{a}{b} = \frac{2c}{2d}$ contradiction

$\square$

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

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- Notice $t > p_i$ for all i.

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- Notice $t > p_i$ for all i. So if t is prime, contradiction.

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- Notice $t > p_i$ for all i. So if t is prime, contradiction.
- If t is not prime then $\exists$ prime factor $q \neq t$ of t

t is composite

[every integer is a product of primes]

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

- Assume that #primes is finite: $p_1, p_2, \dots, p_n$
- Let $t = 1 + \prod_{i=1}^{n} p_i$ (i.e., $1 + p_1 \times p_2 \times \dots \times p_n$)
- Notice $t > p_i$ for all i. So if t is prime, contradiction.
- If t is not prime then $\exists$ prime factor $q \neq t$ of t

  t is composite

  - if $q \neq p_i$ (for all i) ?

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

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- Let $t = 1 + \prod_{i=1}^{n} p_i$ (i.e., $1 + p_1 \times p_2 \times \cdots \times p_n$)
- Notice $t > p_i$ for all i. So if t is prime, contradiction.
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  t is composite

  - if $q \neq p_i$ (for all i), contradiction.

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

- Assume that #primes is finite: $p_1, p_2, \ldots, p_n$
- Let $t = 1 + \prod_{i=1}^{n} p_i$ (i.e., $1 + p_1 \times p_2 \times \cdots \times p_n$)
- Notice $t > p_i$ for all i. So if t is prime, contradiction.
- If t is not prime then $\exists$ prime factor $q \neq t$ of t

  t is composite

  - if $q \neq p_i$ (for all i), contradiction.
  - if $q = p_j$?

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

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  - if $q \neq p_i$ (for all i), contradiction.
  - if $q = p_j$, we know q divides $\prod_{i=1}^{n} p_i$

but then ?

THERE ARE AN INFINITE NUMBER OF PRIMES

(proof by contradiction)

- Assume that #primes is finite: $p_1, p_2, \dots, p_n$
- Let $t = 1 + \prod_{i=1}^{n} p_i$ (i.e., $1 + p_1 \times p_2 \times \cdots \times p_n$)
- Notice $t > p_i$ for all i. So if t is prime, contradiction.
- If t is not prime then $\exists$ prime factor $q \neq t$ of t

  t is composite

  - if $q \neq p_i$ (for all i), contradiction.
  - if $q = p_j$, we know q divides $\prod_{i=1}^{n} p_i$

    but then it can't also divide $1 + \prod_{i=1}^{n} p_i$ (contr.) $\square$

Next:

A variant of proof by contradiction

Prove that the first n odd natural numbers sum to n^2.

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{lccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n \\ & 1 + & 3 + & 5 + & 7 + & \cdots + & ? & ? \end{array}$$

Prove that the first n odd natural numbers sum to n^2.

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

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Sum: 1 4 9 16 ...

so far so good

Prove that the first n odd natural numbers sum to n^2.

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$.

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{lcccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n & \\ & 1 & +\,3 & +\,5 & +\,7 & +\cdots+ & (2n-3) & +\,(2n-1) & = n^2 \end{array}$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$.

We saw the claim is true for small n.

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{ccccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n & \\ & 1 & + 3 & + 5 & + 7 & + \cdots & + (2n-3) & + (2n-1) & = n^2 \end{array}$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$.

We saw the claim is true for small n.

If the claim is false, there must be some smallest number x $(\leq n)$ for which the claim is false.

SMALLEST COUNTEREXAMPLE

Prove that the first n odd natural numbers sum to n^2.

$$\begin{array}{lccccccc} i = & 1 & 2 & 3 & 4 & \cdots & (n-1) & n \end{array}$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2$$

Sum: 1 4 9 16 ...

Suppose not. Then $\sum_{i=1}^{n} 2i-1 \neq n^2$.

We saw the claim is true for small n.

If the claim is false, there must be some smallest number x $(\leq n)$

for which $\sum_{i=1}^{x} 2i-1 \neq x^2$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \; ?$$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \;?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}_{\text{in fact for all } <x}$ for which it is true

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $x-1$ for which it is true

↳ in fact for all $< x$

$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i = x-1} = (x-1)^2$$

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$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \ ?$$

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$$1 + 3 + 5 + \cdots + \overbrace{(2x-3)}^{i=x-1} = (x-1)^2$$

$$\underbrace{1 + 3 + 5 + \cdots + (2x-3)}_{(x-1)^2} + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$+ 2x-1 \neq x^2$$

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \; ?$$

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$$1 + 3 + 5 + \cdots + (2x-3) + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$\underline{(x-1)^2} + 2x-1 \neq x^2$$

$$\underline{x^2 - 2x + 1} + 2x - 1 \neq x^2$$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

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$$(x-1)^2 \qquad + 2x-1 \neq x^2$$

$$x^2 - \cancel{2x} + \cancel{1} \qquad + \cancel{2x} - \cancel{1} \neq x^2$$

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \quad ?$$

if false, then $\exists x$ for which it is false & $\underbrace{x-1}$ for which it is true
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$$1 + 3 + 5 + \cdots + (2x-3) + \overbrace{(2x-1)}^{i=x} \neq x^2$$

$$(x-1)^2 + 2x-1 \neq x^2$$

$$x^2 - \cancel{2x} + \cancel{1} + \cancel{2x} - \cancel{1} \neq x^2$$

contradiction

$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$

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← either this should have been $\neq$

$$1 + 3 + 5 + \cdots + (2x-3) + \overbrace{(2x-1)}^{i = x} \neq x^2$$

← or this should have been $=$

$$(x-1)^2 + 2x - 1 \neq x^2$$

$$\underline{x^2} - \cancel{2x} + \cancel{1} + \cancel{2x} - \cancel{1} \; \underline{\neq x^2}$$

contradiction

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \;?$$

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... which contradicts the smallest counterexample assumption.

$$(x-1)^2 + 2x - 1 \neq x^2$$

$$x^2 \cancel{-2x} \cancel{+1} \cancel{+2x} \cancel{-1} \neq x^2$$

contradiction

$$i = 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad (n-1) \quad n$$

$$1 + 3 + 5 + 7 + \cdots + (2n-3) + (2n-1) = n^2 \; ?$$

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$$(x-1)^2 + 2x - 1 \neq x^2$$

$$x^2 \cancel{-2x} \cancel{+1} \cancel{+2x} \cancel{-1} \neq x^2$$

contradiction

... which contradicts the smallest counterexample assumption, i.e.,

THERE IS NO (SMALLEST) COUNTEREXAMPLE

↳ CLAIM IS TRUE

SMALLEST COUNTEREXAMPLE

recap

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

(case / example)

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

  (case / example)

- prove the claim for smallest instance

  (& prove a smallest instance exists)

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim (case / example)
- prove the claim for smallest instance
- assume the claim is false

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

(case / example)

- prove the claim for smallest instance

- assume the claim is false: then there is a smallest instance, E_i, for which it is false

(smallest counterexample)

SMALLEST COUNTEREXAMPLE recap

- be able to "count" & "order" instances of the claim

  (case / example)

- prove the claim for smallest instance

- assume the claim is false: then there is a smallest instance, E_i, for which it is false

  (smallest counterexample)

- this implies the claim is true for the next smallest instance, E_{i-1}.

SMALLEST COUNTEREXAMPLE

recap

see "well-ordering principle"

- be able to "count" & "order" instances of the claim

  (case / example)

- prove the claim for smallest instance
- assume the claim is false: then there is a smallest instance, E_i, for which it is false

  (smallest counterexample)

- this implies the claim is true for the next smallest instance, E_{i-1}.
- use E_i & E_{i-1} to get a contradiction (to the existence of any counterexample)

Claim: For $n \in \mathbb{Z}$, $n \geqslant 5$, $2^n > n^2$

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

(notice $\left\{ \begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array} \right.$)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

$$\left(\text{notice} \begin{cases} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{cases} \right)$$

- use smallest counterexample

↳ which is ... ?

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

$$\left(\text{notice}\left\{\begin{array}{ccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ 2^n & 1 & 2 & 4 & 8 & 16 & 32 \\ n^2 & 0 & 1 & 4 & 9 & 16 & 25 \end{array}\right.\right)$$

- use smallest counterexample

↳ which is some unknown hypothetical x.

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

notice

n	0	1	2	3	4	5
2^n	1	2	4	8	16	32
n^2	0	1	4	9	16	25

- use smallest counterexample

  ($n = 2, 3, 4$ are not counterexamples)

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  $\hookrightarrow$ why can we? $\rightarrow$ Claim is true for smallest instance ($n=5$)

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)

- use smallest counterexample

  ($n=2,3,4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. → ?

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  ($n=2,3,4$ are not counterexamples)

  why can we? → Claim is true for smallest instance ($n=5$)

- assume $\exists$ smallest counterexample x. $\}$ $\boxed{2^x \leq x^2}$ $(x>5)$

  (what other condition?)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

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  why can we? $\rightarrow$ Claim is true for smallest instance ($n=5$)

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  (& for $y \geq 5$, if $y < x$ then $2^y > y^2$)

Claim: For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

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- use smallest counterexample

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- focus on $x-1$:

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- assume $\exists$ smallest counterexample x. $\boxed{2^x \leq x^2}$ ($x > 5$)

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- focus on $x-1$: $\boxed{2^{x-1} > (x-1)^2}$

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- assume $\exists$ smallest counterexample x. $\Big\}$ $\boxed{2^x \leq x^2}$ $(x>5)$

  (& for $y \geq 5$, if $y<x$ then $2^y > y^2$)

- focus on $x-1$: $\boxed{2^{x-1} > (x-1)^2}$ → combine to get contradiction

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because ?

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and not the smallest case

next?

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

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$$2^x > 2x^2 - 4x + 2$$

$$\boxed{2^x \le x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

$\boxed{2^x \leq x^2}$

because x is a counterexample

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$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

if $x^2 - 4x + 2 \geq 0$

then ?

$\boxed{2^x \le x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample

and not the smallest case

!

$$2^{x-1} > x^2 - 2x + 1$$

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if $x^2 - 4x + 2 \geq 0$ we will get a contradiction

$$\boxed{2^x \leq x^2}$$

because x is a counterexample

$$\boxed{2^{x-1} > (x-1)^2}$$

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!

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$$2^x > x^2 + (x^2 - 4x + 2)$$

if $x^2 - 4x + 2 \geqslant 0$ we will get a contradiction

$$\updownarrow$$

$$(x-2)\cdot(x-2) \geqslant 2$$

true for $x \geqslant 4$

$\boxed{2^x \leq x^2}$

because x is a counterexample

$\boxed{2^{x-1} > (x-1)^2}$

because x is the smallest counterexample and not the smallest case

$$2^{x-1} > x^2 - 2x + 1$$

$$2^{x-1} \cdot 2 > 2x^2 - 4x + 2$$

$$2^x > 2x^2 - 4x + 2$$

$$2^x > x^2 + (x^2 - 4x + 2)$$

!

if $x^2 - 4x + 2 \geq 0$ we will get a contradiction

$\Updownarrow$

$(x-2)\cdot(x-2) \geq 2$

true for $x \geq 4$

We have assumed $x > 5$

$\square$

conclusion

For $n \in \mathbb{Z}$, $n \geq 5$, $2^n > n^2$

FIBONACCI NUMBERS

FIBONACCI NUMBERS

$F_0 = 1 \qquad F_1 = 1$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$F_0 = 1$ $F_1 = 1$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

FIBONACCI NUMBERS

for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

FIBONACCI NUMBERS

for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

etc

FIBONACCI NUMBERS

for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

$F_2 = 2$

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$F_4 = 5$

$F_5 = 8$

$F_6 = 13$

etc

Claim: for $n \in \mathbb{Z}$, $n \geqslant 0$, $F_n \leq 1.7^n$

FIBONACCI NUMBERS

for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

$\boxed{F_0 = 1}$ $\boxed{F_1 = 1}$

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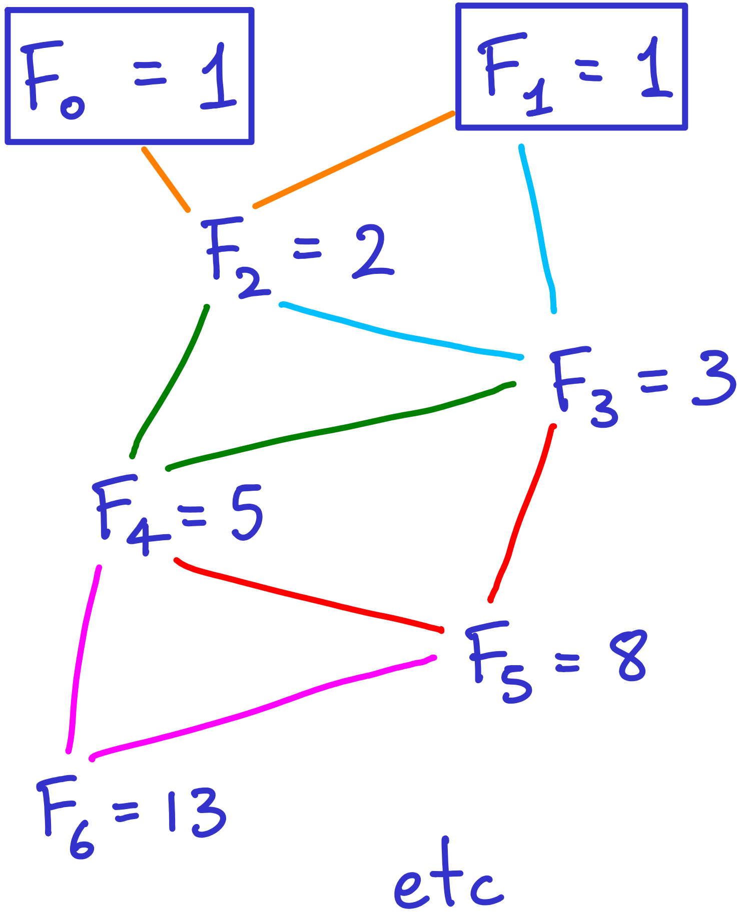

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slight hiccup ?

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so why not use F_{x-2} also ?

(why not F_{x+1} ?)

FIBONACCI NUMBERS

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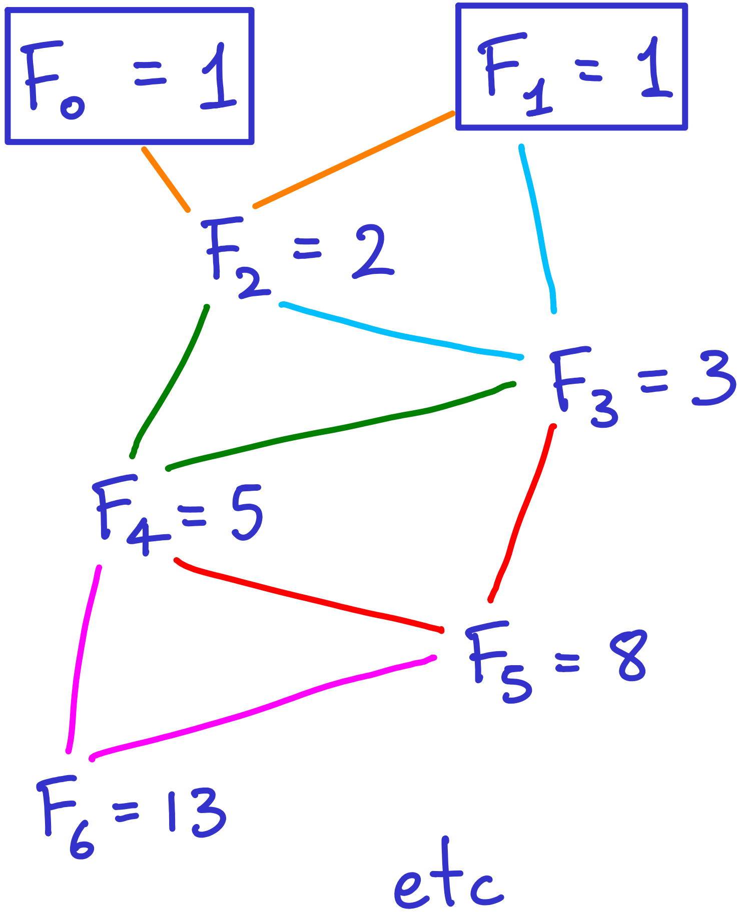

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it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geq 2$

$\hookrightarrow$ is $F_0 \leq 1.7^0$?

FIBONACCI NUMBERS

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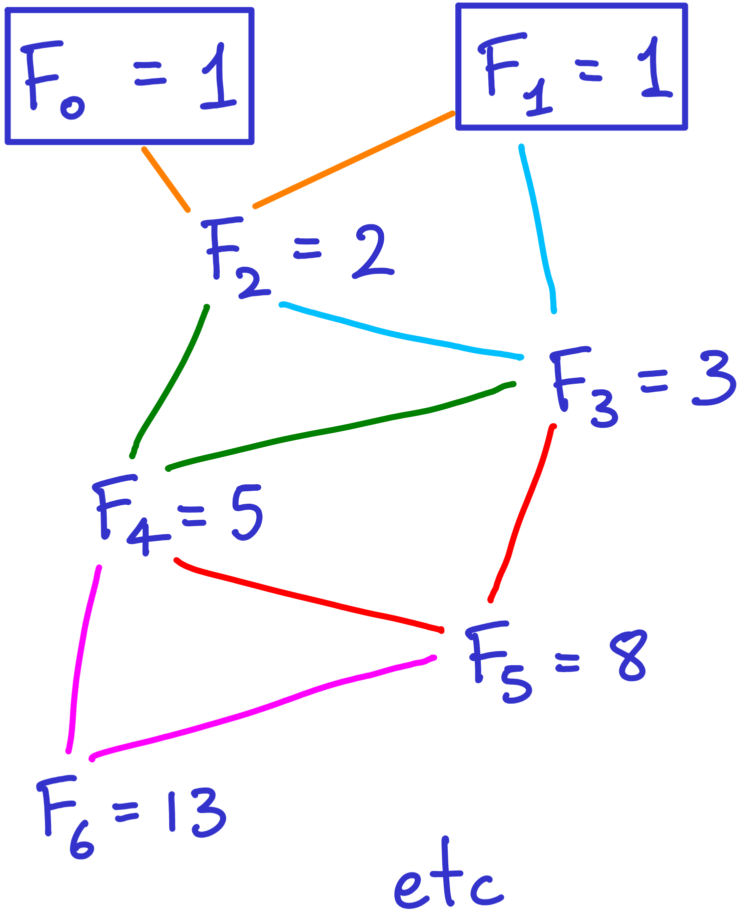

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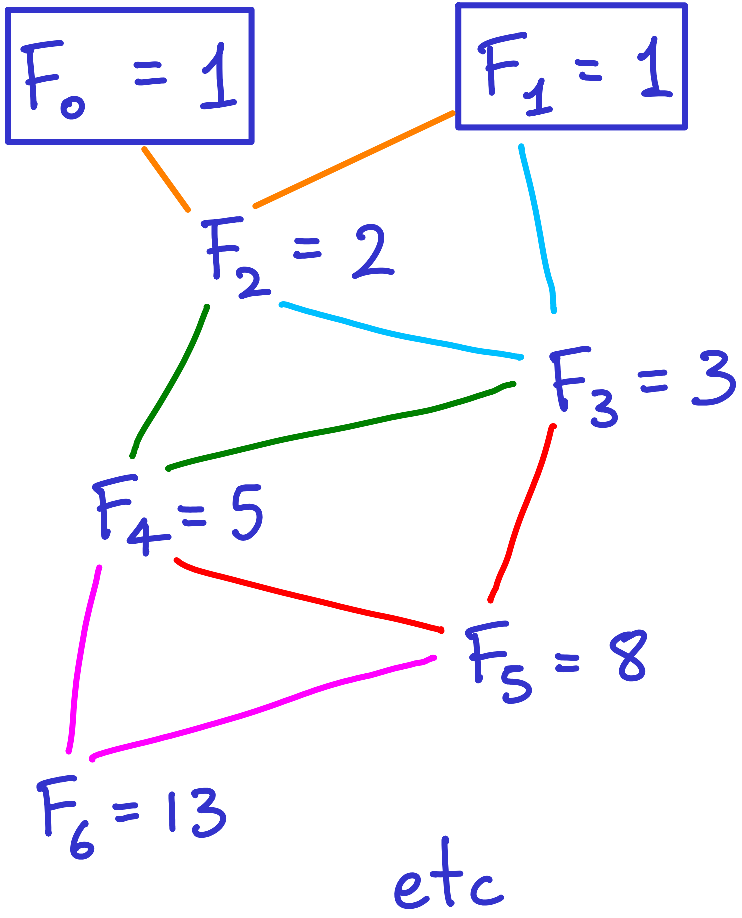

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it will be hard to use only F_x & F_{x-1}

so why not use F_{x-2} also: assume $x \geq 2$

$\hookrightarrow$ is $F_0 \leq 1.7^0$? yes. Is $F_1 \leq 1.7^1$? yes. OK!

$F_0 = F_1 = 1$ // for $n \geqslant 2$, $F_n = F_{n-1} + F_{n-2}$

Claim: for $n \in \mathbb{Z}$, $n \geqslant 0$, $F_n \leq 1.7^n$

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$(x \geqslant 2)$ $F_y \leq 1.7^y$ for $y < x$

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next?

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we can now say: $F_x = F_{x-1} + F_{x-2}$ ...

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$= 1.7^{x-2} \cdot (1.7 + 1)$

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$< 1.7^{x-2} \cdot (1.7)^2$ $[1.7^2 = 2.89]$

next?

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$$= 1.7^{x-2} \cdot (1.7 + 1)$$

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$$< 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89]$$

$$= 1.7^x$$

so?

$F_0 = F_1 = 1$ // for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

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& we can safely assume $F_y \leq 1.7^y$ for $y < x$

we can now say: $F_x = F_{x-1} + F_{x-2} \leq 1.7^{x-1} + 1.7^{x-2}$

$$
\begin{aligned}
&= 1.7^{x-2} \cdot (1.7 + 1) \\
&= 1.7^{x-2} \cdot 2.7 \\
&< 1.7^{x-2} \cdot (1.7)^2 \quad [1.7^2 = 2.89] \\
&= 1.7^x
\end{aligned}
$$

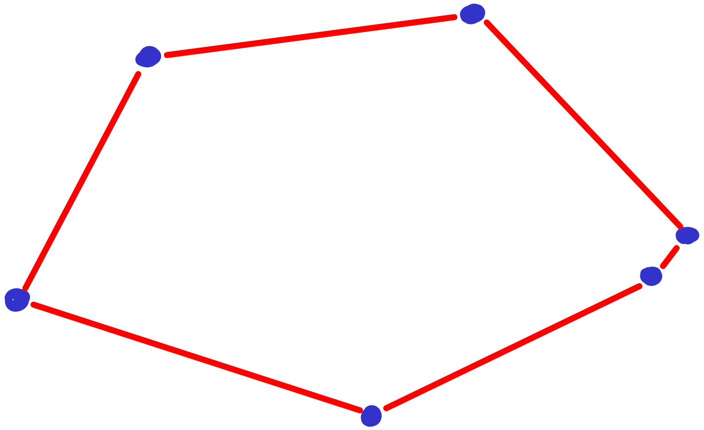
so $F_x < 1.7^x$

CONTRADICTION

$\square$

geometry time

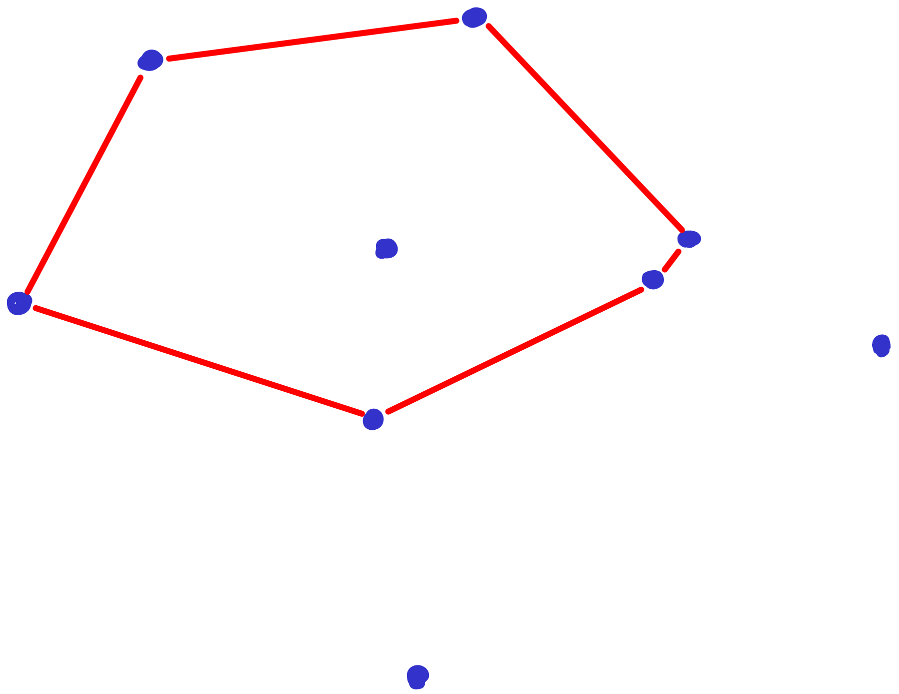
6 points in convex position.



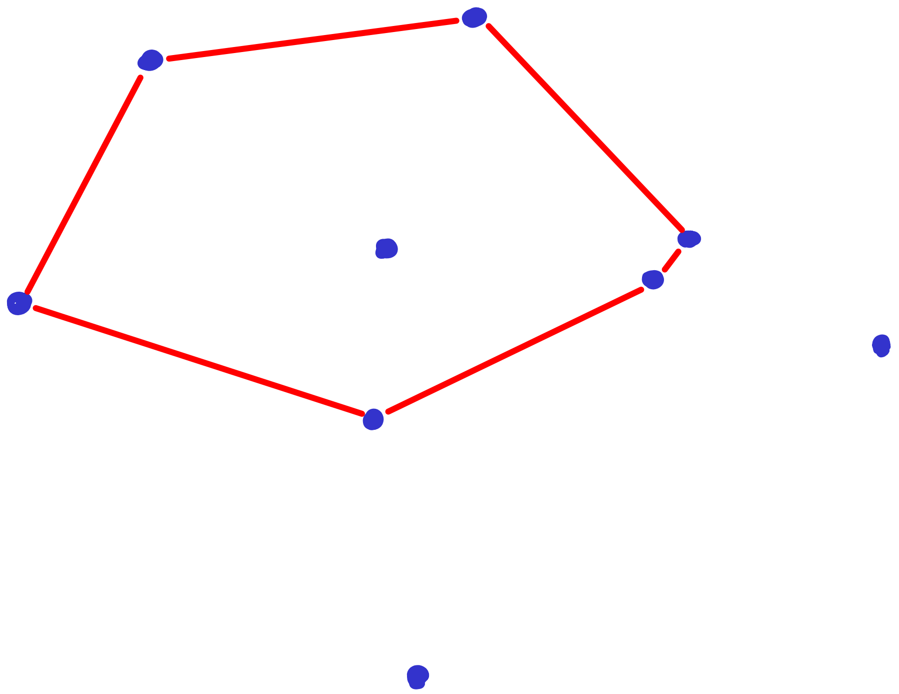
This is in 2D, aka "the plane".

x,y coordinates are real numbers, so our point set is in $\mathbb{R}^2$

still 6 points in convex position.

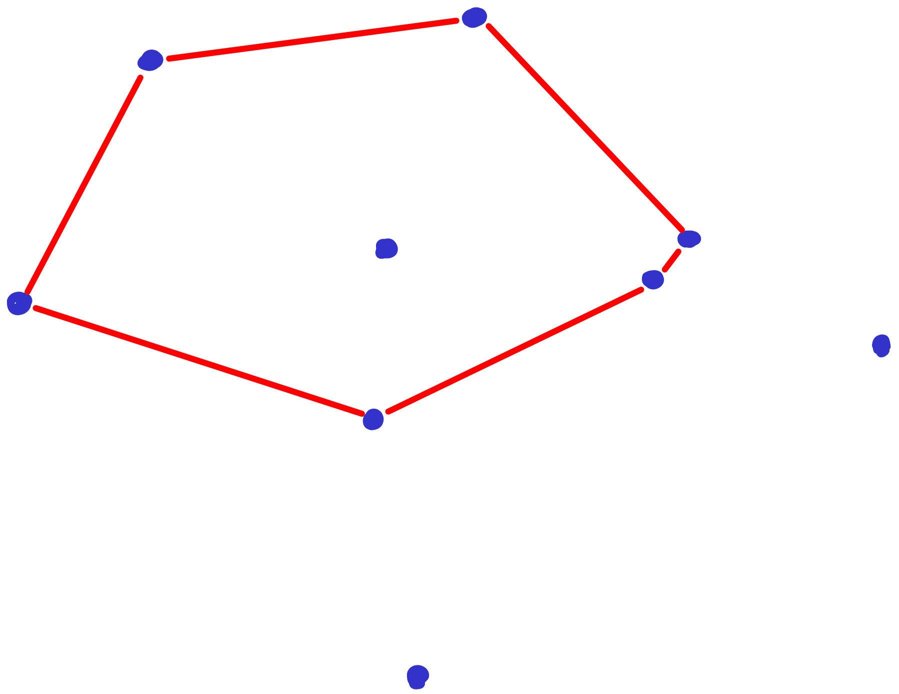


Theorem: in $\mathbb{R}^2$, every set of ≥ 17 points w/ no 3 on a line has 6 points in convex position.



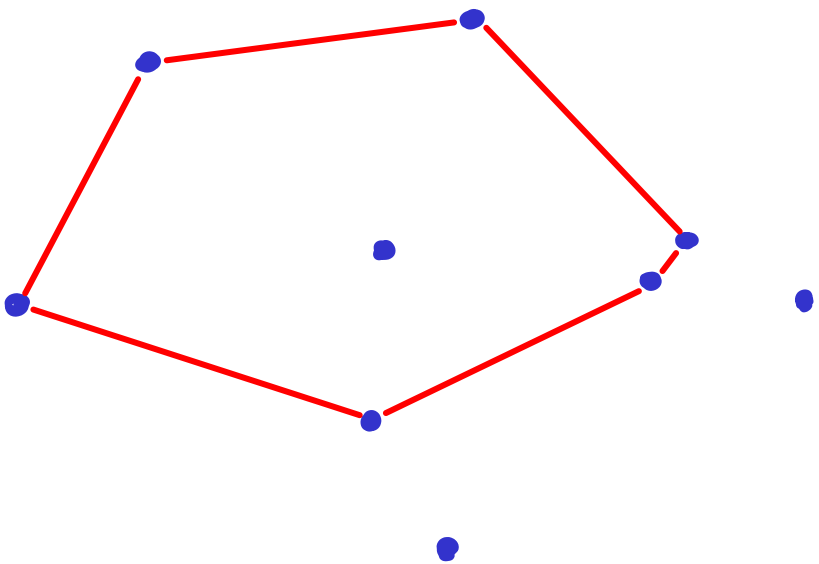
Theorem: in $\mathbb{R}^2$, every set of $\geqslant 17$ points w/ no 3 on a line
has 6 points in convex position.

"has a hexagon"
(not necessarily regular)

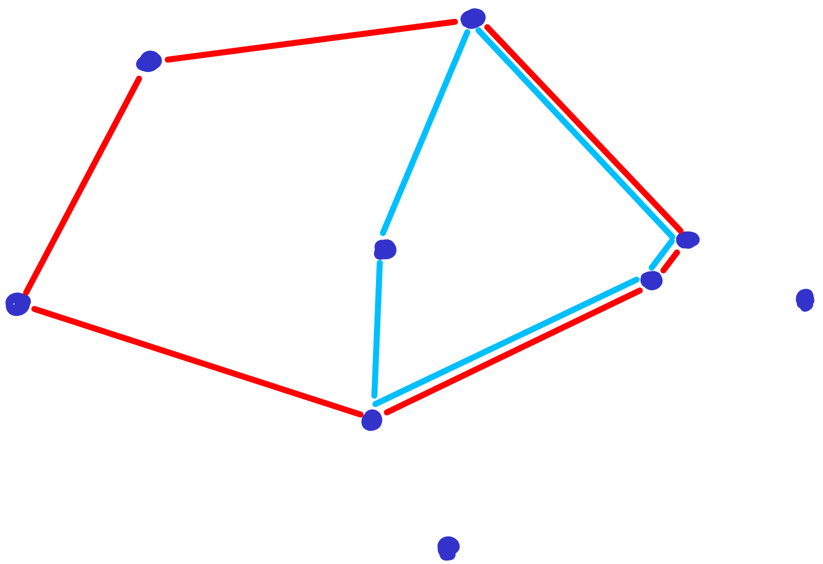


Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,

if P has 6 points forming a hexagon...



Claim: in $\mathbb{R}^2$, given a set of points P w/ no 3 on a line,
if P has 6 points forming a hexagon
then P has 5 points forming an empty pentagon.



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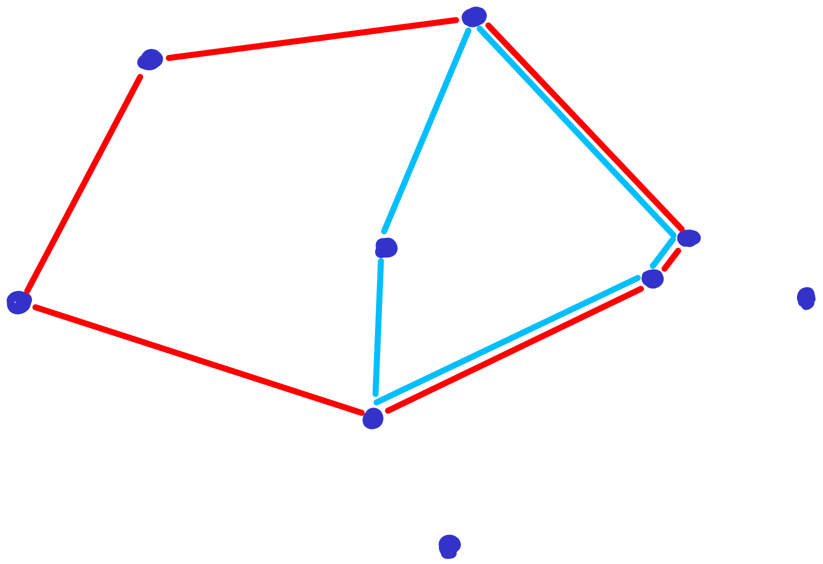


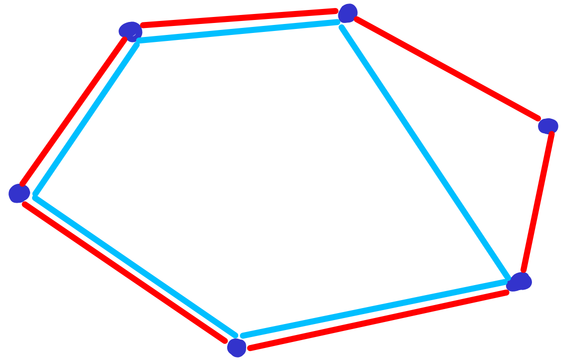
Stronger claim:

Every hexagon contains an empty pentagon

Claim: Every hexagon H contains an empty pentagon

Trivial example :

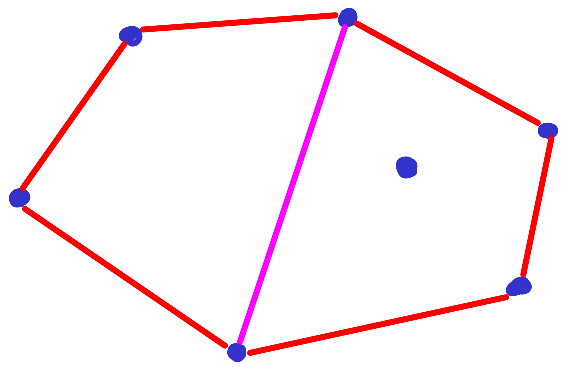
- if H is empty, DONE.



Claim: Every hexagon H contains an empty pentagon

Trivial examples:

- if H is empty, DONE.
- if H contains exactly 1 point, "split" H



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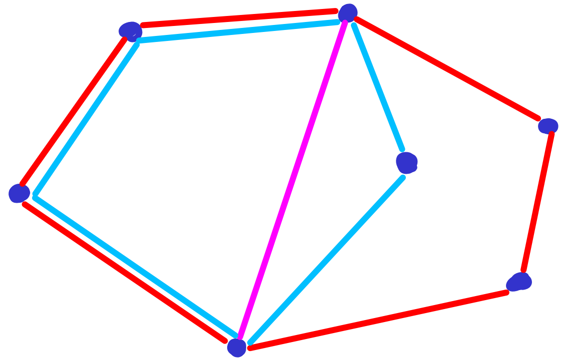

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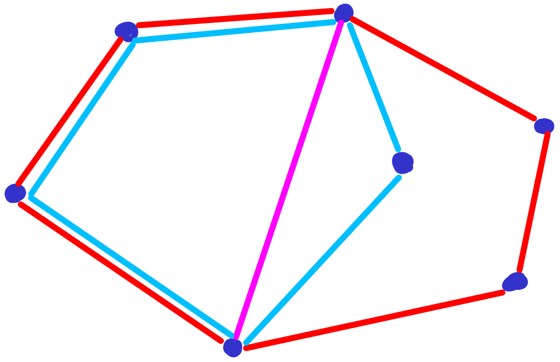

- We can order hexagons by #points inside.

Claim: Every hexagon H contains an empty pentagon

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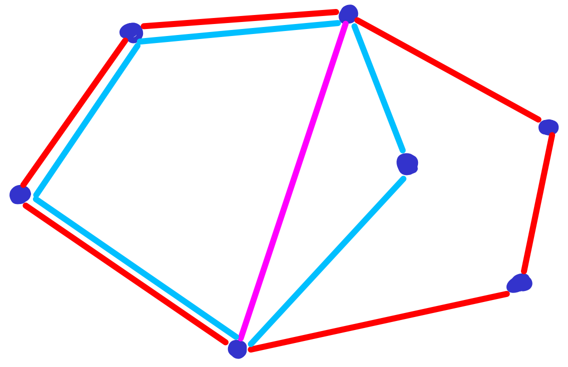

- We can order hexagons by # points inside.
- If claim is false there must be a smallest counterexample

Claim: Every hexagon H contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon H containing min # pts, for which claim is false.

- shown: if H contains ≤ 1 points, DONE $\rightarrow$ so assume ≥ 2 pts inside.

Claim: Every hexagon H contains an empty pentagon

Proof by smallest counterexample

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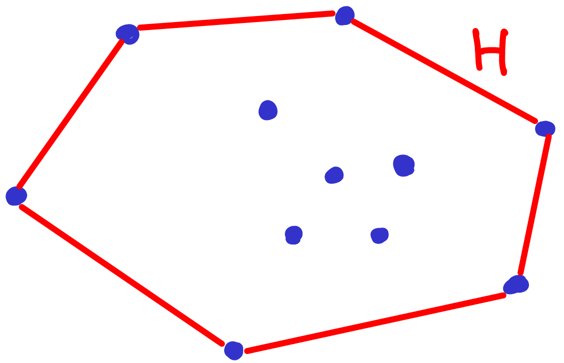

hypothetical smallest counterexample

Claim: Every hexagon H contains an empty pentagon

Proof by smallest counterexample

Choose a hexagon H containing min #pts, for which claim is false.

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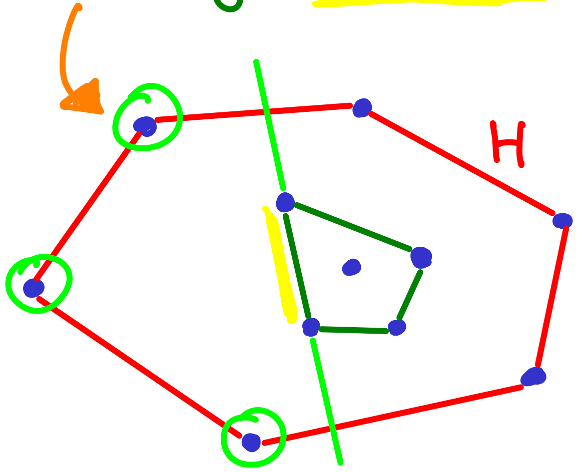
- if any "extreme segment" of interior points "isolates" 3 points of H ...



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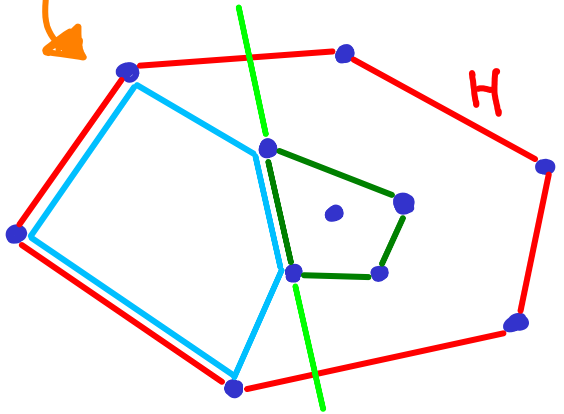

this wasn't a counterexample,

CONTRADICTION

Claim: Every hexagon H contains an empty pentagon

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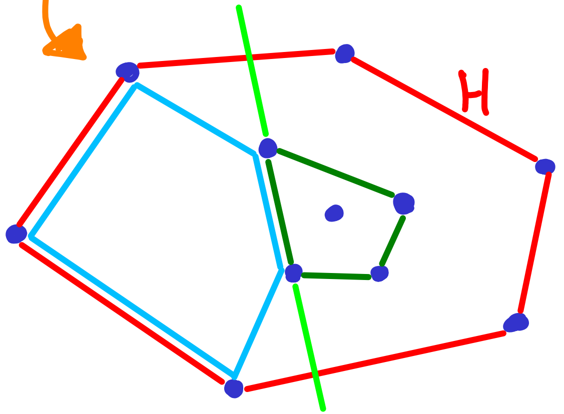
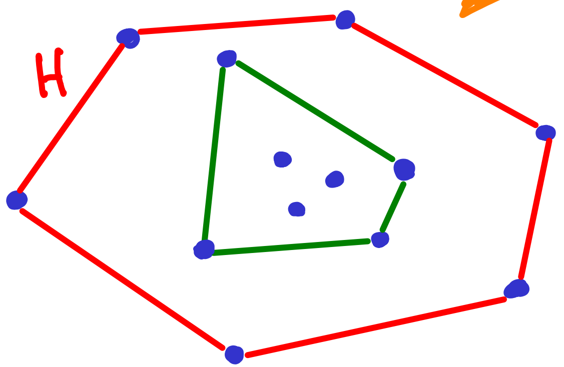
↳ so every such segment isolates 1 or 2 points.



{invalid H



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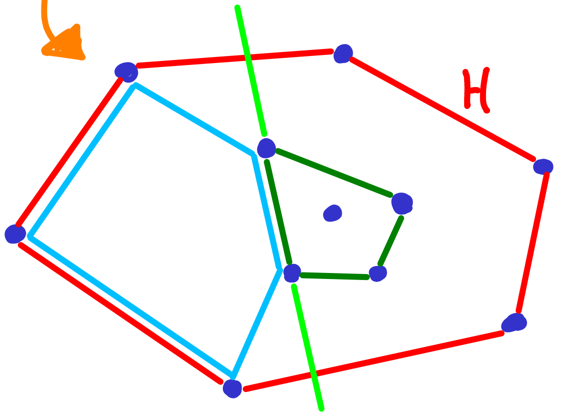
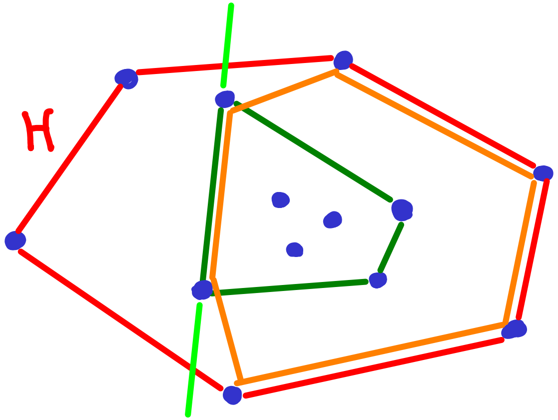


↳ so every such segment isolates 1 or 2 points.

↳ use one segment & form a hexagon H'

(why?)



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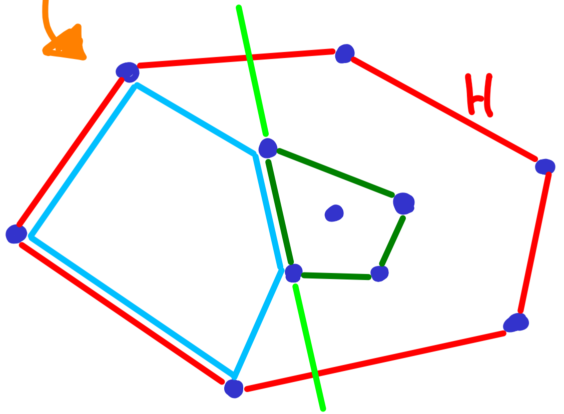
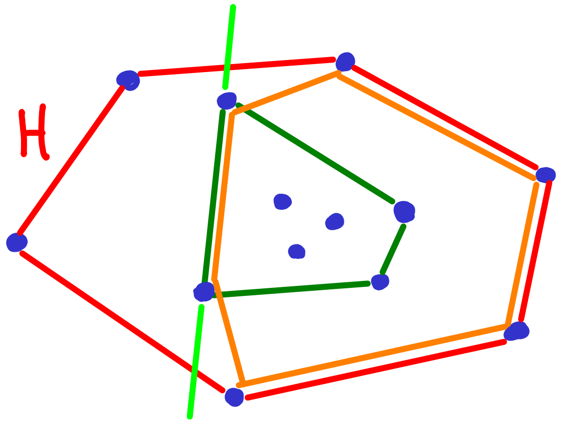


↳ use one segment & form a hexagon H' containing fewer points than H.

(so?)



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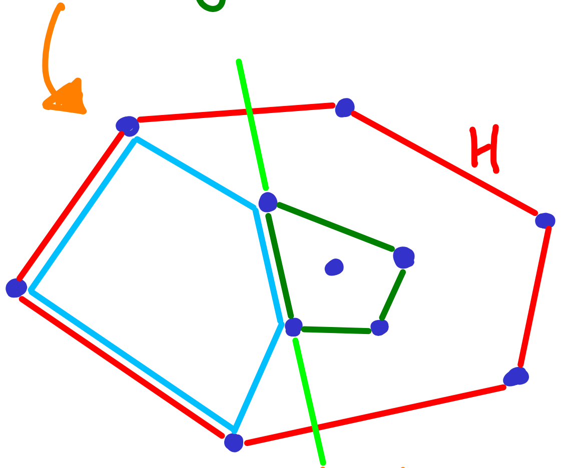

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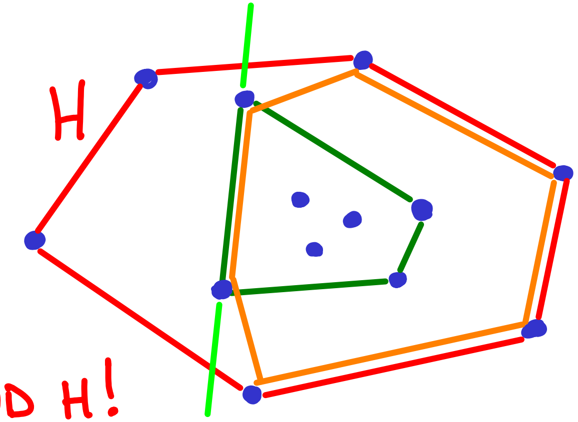

If H is smallest counterexample, claim is true for H' AND H! $\square$

The "smallest counterexample" method is useful and elegant, and essentially the same as another extremely useful method:

INDUCTION

proof by INDUCTION

explained quickly

via conversion from smallest counterexample

To learn Induction from scratch, see other set of notes.

proof by INDUCTION

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)

proof by INDUCTION

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)

like proof by smallest counterexample,

(2) focus on two "neighboring" cases [call them $n-1$ & n]

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like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)

like proof by smallest counterexample,

(2) focus on two "neighboring" cases [call them n-1 & n]

"unlike" proof by smallest counterexample,

(3) show that if the claim is true for n-1 then it is true for n } $A \to B$

proof by INDUCTION

like proof by smallest counterexample,

(1) prove your claim for a base case (should be ~easy)

like proof by smallest counterexample,

(2) focus on two "neighboring" cases [call them n-1 & n]

"unlike" proof by smallest counterexample, ... which proves $(A \wedge \neg B) = F$

which is the same

(3) show that if the claim is true for n-1 then it is true for n $\Big\}$ $A \rightarrow B$