

Non-planar graphs
(can't redraw) K_{S} $K_{3,3}$

 K_S

A graph is non-planar if and only if it "contains" a K3,3 or K5

FYI

Every planar graph can be drawn without crossings. In fact the edges can be drawn straight as well.

EULER FORMULA for planar connected graphs: $V - E + F = 2$

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EULER FORMULA for planar connected graphs: $V - E + F = 2$ Proof by induction on number of faces: Base case > $F = 1$ > G is a free > $V = E + 1$ so $(E+1) - E + 1 = 2 \sqrt{2}$

EULER FORMULA for planar connected graphs:
$$
V-E+F=2
$$

Proof by induction on number of faces:
Base case \rightarrow F=1 \rightarrow G is a tree \rightarrow V=E+1
so (E+1)-E+1 = 2

Given
$$
G=(V,E)
$$
 $\omega / F > 1$ faces,
remove an edge e between 2 faces, $f, 2, f_2, \dots$

EULER FORMULA for planar connected graphs: $V - E + F = 2$ Proof by induction on number of faces: Base case > $F = 1$ > G is a free > $V = E + 1$ so $(E+1) - E + 1 = 2 \sqrt{2}$

Given
$$
G=(V,E)
$$
 $\omega / F > 1$ faces,
remove an edge e between 2 faces, $\frac{\rho}{1} \& \frac{\rho}{2}$.
Either $\frac{\rho}{1}$ or $\frac{\rho}{2}$ is a bounded face,
so e is on a cycle (e is not a cut edge).

EULER FORMULA for planar connected graphs: $V - E + F = 2$ Proof by induction on number of faces: Base case > $F = 1$ > G is a free > $V = E + 1$ so $(E+1) - E + 1 = 2 \sqrt{2}$

Given
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remove an edge e between 2 faces, $f, 2, f_2$.
Either f_1 or f_2 is a bounded face,
so e is on a cycle (e is not a cut edge)
 $\int G-e$ is connected $\& f_1, f_2$ merge

$$
\frac{f_1}{2}
$$

EULER FORMULA for planar connected graphs: $V - E + F = 2$ Proof by induction on number of faces: $\frac{1}{\sqrt{2}}$ Base case > $F = 1$ > G is a free > $V = E + 1$ so $(E+1) - E + 1 = 2$ Given $G=(V,E)$ w/ $F>1$ faces, $\frac{f_1}{2}$ remove an edge e between 2 faces, $f_1 \times f_2$. E ither f_1 or f_2 is a bounded face, so e is on a cycle (e is not a cut edge) G G e is connected & f, f, merge: $V(E-1)$ (F-1)

EULER FORMULA for planar connected graphs: $V - E + F = 2$ Proof by induction on number of faces: Lega Base case > $F = 1$ > G is a free > $V = E + 1$ so $(E+1)-E+1 = 2$ Siven G=(V,E) w/ $F > 1$ taces,
remove an edge e between 2 faces, $f_1 \& f_2$. Given $G=(V,E)$ w/ $F>1$ faces, E ither f_1 or f_2 is a bounded face, so e is on a cycle (e is not a cut edge) hypothesis $G-e$ is connected & f_1, f_2 merge: $V-(E-1)+(E-1)=2$

EULER FORMULA for planar connected graphs:
$$
V - E + F = 2
$$
 \nProof by induction on number of faces: \nBase case \rightarrow F = 1 \rightarrow G is a tree \rightarrow V = E + 1 \nso (E + 1) - E + 1 = 2 \n\nGiven G = (V, E) \omega / F > 1 \nfaces, \nremove an edge e between 2 faces, $\ell_1 \& \ell_2$. \nEither ℓ_1 or ℓ_2 is a bounded face, \nso e is on a cycle (e is not a cut edge) \nbypethesis \nG = e is connected 8. ℓ_1, ℓ_2 merge: $\sqrt{-(E-1) + (F-1)} = 2$ \nNote that this also holds for multiplying \nO\n\n ℓ_1

Euler formula
$$
V-E+F=2
$$

$$
\begin{array}{ll}\n\hline\nV-E+F=2 & applies-to-any connected planar & in-fact, +-convex polyhedra) \\
\hline\nInduction on-faces: & F=1 & \qquad \qquad E=V-1\n\end{array}
$$
\n
$$
F \geq 1 : \text{tree.}
$$
\nRemove an edge between 2 faces.

\nRunning connected.

\n
$$
F \rightarrow F-1 \qquad E \rightarrow E-1
$$

Euler formula
$$
V-E+F=2
$$

$V-E+F=2$	applies -lo any connected planar	in-fact, +e convex polyhedra			
Induction on faces:	Induction on vertices	by projection			
$F=1$	or	if: tree.	$V=1$	for	only loops
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$F \times F$	2 faces.	2 faces.			
Remains connected.	$F \rightarrow F-1$	$E \rightarrow E-1$			

Euler formula
$$
V-E+F=2
$$

$V-E+F=2$	applies $\pm b$	any connected planar	in \pm act, \pm convex polyhedra	
Induction on \pm aces:	Juduction on \pm vertices			
$F=1$:\n $\sqrt{1}$:\n \pm ree.	$V=1$	$\sqrt{1}$	only loops
$F=f$?	$\sqrt{2}$?	1
$F \geq V-1$	$\sqrt{2}$	1		
Remove an edge between 2 faces.	1	1		
Comning connected.	1	1		
Remains connected.	1	1		
$F \rightarrow F-1$	$E \rightarrow E-1$	1		

Euler formula
$$
V-E+F=2
$$

$V-E+F=2$	applies -lo any connected planar	in- $\frac{1}{4}act$, +o convex polyhedra
Induction on -faces: F=1 : $\frac{1}{2}$: tree. F=1 : $\frac{1}{2}$: tree. F>1 : tree. F>2 for the $\frac{1}{2}$ and $\frac{1}{2}$ for the $\frac{1}{2}$ and $\frac{1}{2}$ for the $\frac{1}{2}$		

Euler formula
$$
V-E+F=2
$$

$V-E+F=2$	applies $\pm b$	any connected planar	in- \pm act, \pm convex polyhedra.
Induction on -faces:	Induction on vertices	Subc \pm on on edges	
$F=1$	\therefore \uparrow : tree.	$V=1$	Subc \pm on on edges
$F>2$	Suppose \pm or edge between \pm vertices		
\uparrow ?	\vee or \uparrow is a vertex		
\uparrow ?	Contract edge $\frac{x \neq y}{z \neq 0}$	$E \geq 0$	contract are the same.
Remains connected.	Subc \pm or \pm are the same.		
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\pm or \pm are the same.			
\pm or \pm are the same.			
\pm or <math< td=""></math<>			

$$
use the Euler formula V-E+F=2
$$

to show that a connected plane graph has E < 3V-6 for V>3
Not allowed: V
$$
use the Euler formula V-E+F=2
$$

\n
$$
to show that a connected plane graph has E \le 3V-6
$$

\nEvery edge belongs to 1 or 2 faces
$$
\sum_{all faces} e \le 2E
$$

Use the Euler formula
$$
V - E + F = 2
$$

\nHow that a connected plane graph has $E \leq 3V - 6$

\nEvery edge belongs to 1 or 2 faces all faces

\nEvery face has $\frac{1}{2}$ edges (for $V > 3$) $\sum_{all \text{ faces}} e \geq 3F$

Use the Euler formula
$$
V - E + F = 2
$$

\nto show that a connected plane graph has $E \leq 3V - 6$

\nEvery edge belongs to 1 or 2 faces all faces

\nEvery face has $\frac{1}{2}S^3$ edges (for V)>3) $\sum_{all \text{ faces}} e \geq 3F$

\nEvery face has $\gg 3$ edges (for V)>3) $\sum_{all \text{ faces}} e \gg 3F$

use the Euler formula
$$
V - E + F = 2
$$

\nto show that a connected plane graph has $E \le 3V - 6$
\nEvery edge belongs to 1 or 2 faces $\sum_{all faces} e \le 2E$
\nEvery face has ≥ 3 edges (for V>3) $\sum_{all faces} e \ge 3F$
\nE-F=V-2

Use the Euler formula
$$
V - E + F = 2
$$

\nto show that a connected plane graph has $E \leq 3V - 6$

\nEvery edge belongs to 1 or 2 faces all faces

\nEvery face has $\frac{1}{2}$ edges (for $V > 3$) $\sum_{all faces} e \geq 3F$

\nE- $F = V - 2$

\nE- $\frac{2E}{3} \leq V - 2$

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Use the Euler formula
$$
V - E + F = 2
$$

\nto show that a connected plane graph has $E \leq 3V - 6$

\nEvery edge belongs to 1 or 2 faces all faces

\nEvery face has $\frac{1}{2}S$ edges (for $V > 3$) $\sum_{all faces} e \geq 3F$

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\nE- $\frac{2E}{3} \leq V - 2$

\nE- $\frac{3E}{2} \leq V - 4$

\nExperiments:

\nExample 21. The sum of the following matrices are not provided.

$E \le 3V-6$

 $E \le 3V-6$
10 $\le 15-6$
!!!

Not planar

 $E \le 3V-6$ $10 \le 15 - 6$ $\frac{1}{1}$

$E \le 3V-6$

 $E \le 3V-6$ $10 \le 15 - 6$ $\frac{1}{1}$

 $E \le 3V-6$ $9 \le 18-6$ ok!

 $E \le 3V-6$ $10 \le 15 - 6$ $\frac{1}{1}$

 $E \leq 3V-6$ $9 \le 18-6$ ok!

not iff All planar graphs have $E\leq 3v-6$ Some non-planar graphs can too

What if G has no triangles? $V - E + F = 2$

V-E+F=2 What if G has no triangles?
\nEvery edge belongs to 1 or 2 faces
$$
\sum_{all faces} e \le 2E
$$

\nEvery face has $\frac{24}{25}$ edges (for V)<4) $\sum_{all faces} e \ge 4F$
\n=

V-E+F=2 What if G has no triangles?
\nEvery edge belongs to 1 or 2 faces
$$
\sum_{all \text{ faces}} e \le 2E
$$

\nEvery face has $\sum_{real \text{ edges}} f$
\nE-F=V-2
\nE=E=V-2
\nE=2V-4
\nInstead of \$3V-6

TRIANGULATIONS

Assume outer face is a triangle $\frac{E=3V-6}{Why?}$

E=3V-6
Why? Assume outer face is a triangle Every edge belongs to $\frac{1}{2}$ faces $\sum_{all \text{ faces}} e \times 2E$
Every face has \bar{x} 3 edges (for $\sqrt{3}$) $\sum_{all \text{ faces}} e \times 3F$)

 $\frac{E=3V-6}{Why?}$ Assume outer face is a triangle $V-E+F=2$ Every edge belongs to $\frac{1}{2}$ faces $\sum_{all \text{ faces}} e \times 2E$
Every face has $\overline{3}$ faces (for $V>3$) $\sum_{all \text{ faces}} e \times 3F$)

$$
E-F=V-2
$$

E=3V-6
Why? Assume outer face is a triangle $V - E + F = 2$ Every edge belongs to $\frac{1}{2}$ faces $\sum_{all \text{ faces}} e \times 2E$
Every face has \overline{x} 3 edges (for $v > 3$) $\sum_{all \text{ faces}} e \times 3F$)

$$
E-F=\sqrt{-2}
$$

$$
E-\frac{2E}{3}=\sqrt{-2}
$$

$$
E=3\sqrt{-6}
$$

What is the average degree of a triangulation?

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 \overrightarrow{v} $\sum_{i=1}^{V} d(v_i)$

What is the average degree of a triangulation?
 $\frac{1}{V}\sum_{i=1}^{V}d(v_i) = \frac{1}{V}\cdot 2E$

What is the average degree of a triangulation?
 $\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6$

E=3V-6
What is the average degree of a triangular?
\n
$$
\frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \le 6
$$

\nS Every triangular has a vertex is/ degree ≤ 5
\nS Immediately applies to any planar graph
\n(euser edges)

What is the average degree of a triangulation?
 $\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6$ $E = 3V - 6$ 4 Every planar graph has a vertex w/ degree <5

$$
\frac{E=3V-6}{V}\frac{What}{\frac{1}{V}\sum_{i=1}^{V}d(v_i)} = \frac{1}{V}\cdot 2E = \frac{6V-12}{V} \leq \frac{6}{V}
$$

G Every planar graph has a vertex w/ degree
$$
\leq
$$

E=3V-6 What is the average degree of a triangular?
$$
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \le 6
$$

4 Every planar graph has a vertex w/ degree <5

Can we find many low-degree vertices?
$$
\rightarrow
$$
 not if "low" = 5.
Can we find many low-degree vertices? \rightarrow what if "low" = 8?
E=3V-6 What is the average degree of a triangular?
\n
$$
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \le 6
$$

\n~~S~~ Every planar graph has a vertex ω degree ≤ 5

Can we find many low-degree vertices?
$$
\rightarrow
$$
 not if "low" = 5.
\n \rightarrow $\frac{v}{2}$ degree - 8

E=3V-6
\nWhat is the average degree of a triangular?
\n
$$
\frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \le 6
$$

\nEvery planar graph has a vertex w/ degree ≤ 5
\nCan we find many low-degree vertices? \rightarrow not if "low" = 5.
\nSay you had $\frac{V}{2}$ vertices w/ degree ≥ 7

Example 2016

\nWhat is the average degree of a triangular function?

\nUsing the following equations:

\n
$$
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} \leq 6
$$
\nUsing plane graph has a vertex by degree 65.

\nCan we find many low-degree vertices?

\nSo, you had $\frac{V}{2}$ vertices of degree 3/8.

\nSo, you had $\frac{V}{2}$ vertices of degree 3/8.

\nSo, you have of $\frac{V}{2}$ vertices of degree 3/8.

\nSo, you have of $\frac{V}{2}$ of them.

Example 3V-6

\nWhat is the average degree of a triangular?

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\nHow,
$$
\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \leq 6
$$

\nCan we find many low-degree vertices?

\nHow, i "low" = 5?

\nSay you had $\frac{V}{2}$ vertices of the square and the square is $\frac{V}{2}$ vertices of the square and the square is $\frac{V}{2}$ vertices of the square.

\nThus, $9 \cdot \frac{V}{2}$ is $\frac{V}{2}$ is <

E=3V-6
\nWhat is the average degree of a triangular?
\n
$$
\frac{1}{V} \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} \le 6
$$

\n
\n*Can we find many low-degree vertices?* \rightarrow not if "low" = 5.
\n
\n*Can we find many low-degree vertices?* \rightarrow not if "low" = 8?
\n
\nSay you had $\frac{V}{2}$ vertices \rightarrow 8
\n \rightarrow 8
\n $\sum_{j>9} d(v_i) \rightarrow 9 \cdot \frac{V}{2}$
\n \rightarrow 8
\n \rightarrow