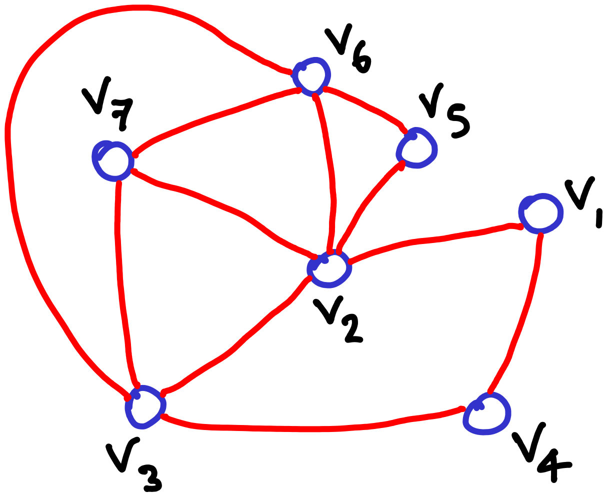
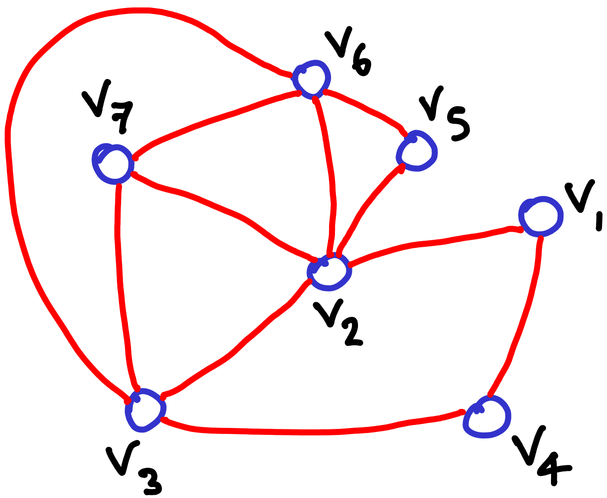
PLANE GRAPH

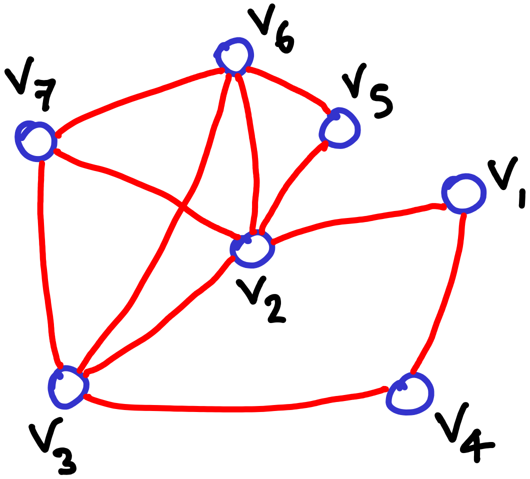
no crossings



PLANE GRAPH

no crossings



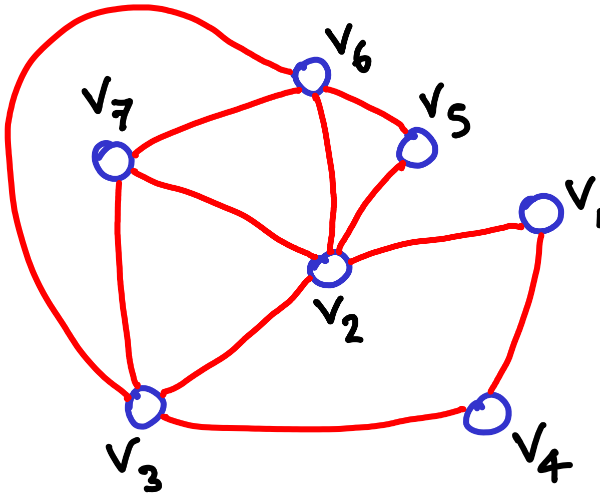


PLANAR GRAPH

can redraw without crossings

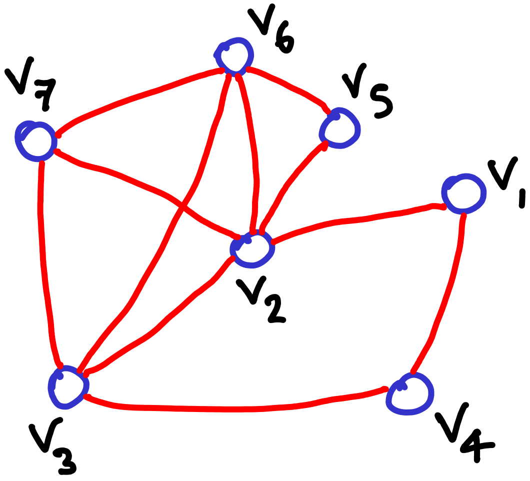
PLANE GRAPH

no crossings

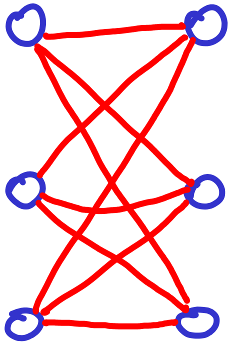


PLANAR GRAPH

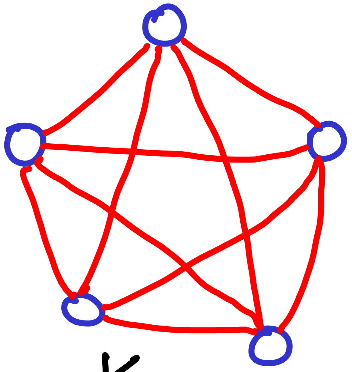
can redraw without crossings



Non-planar graphs
(can't redraw)



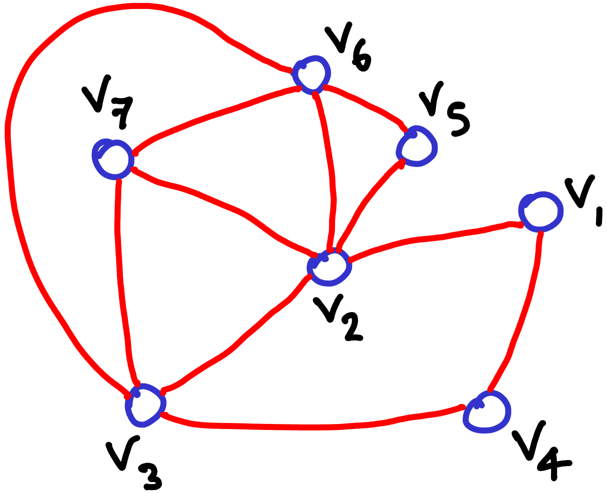
$K_{3,3}$

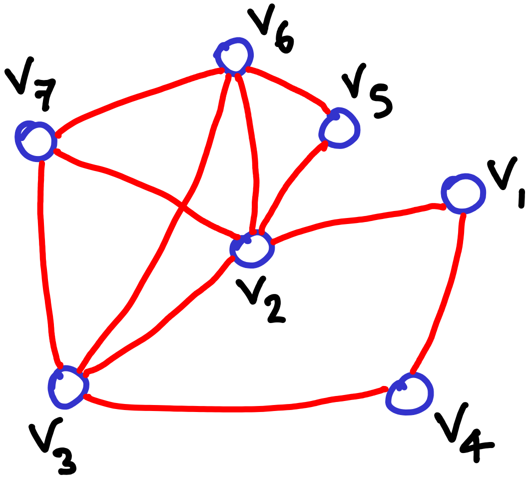


K_5

PLANE GRAPH

no crossings

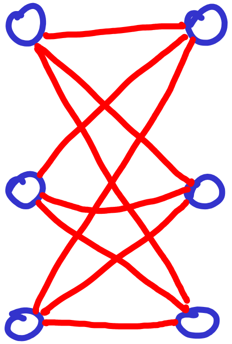




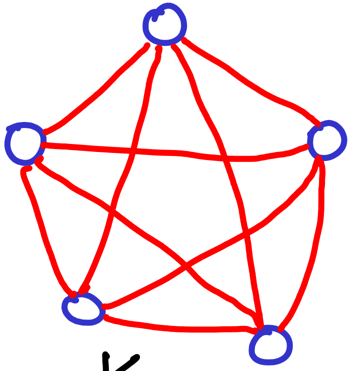
PLANAR GRAPH

can redraw without crossings

Non-planar graphs (can't redraw)



$K_{3,3}$



K_5

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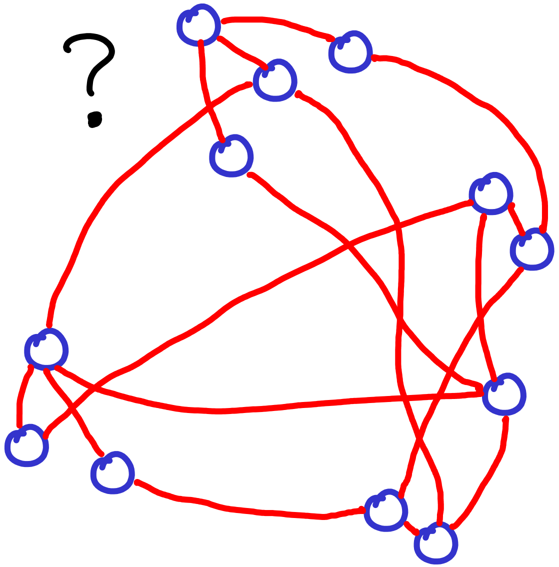
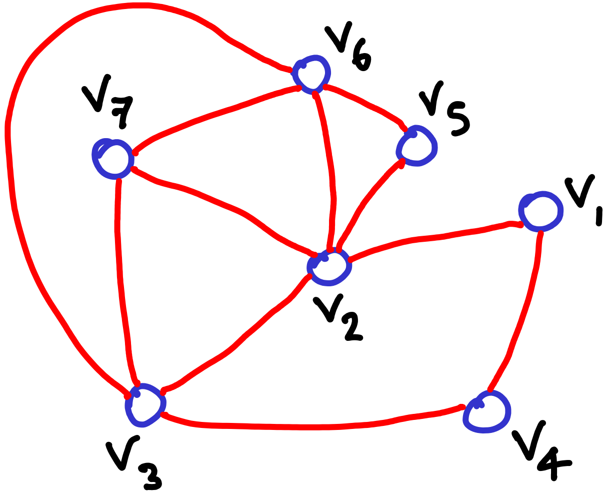
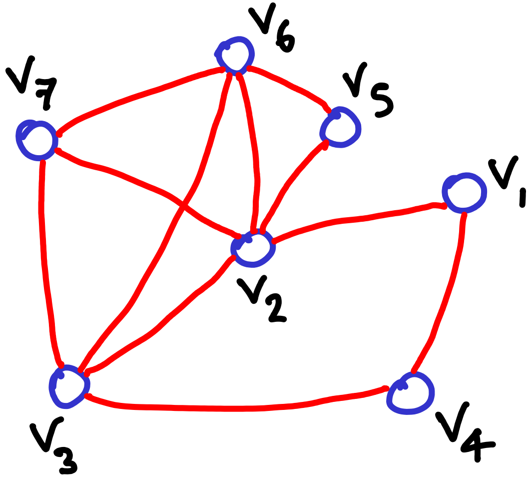


PLANE GRAPH

no crossings

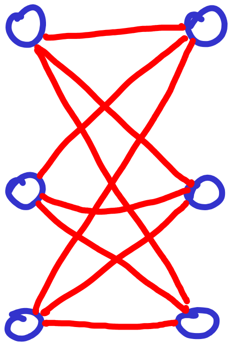




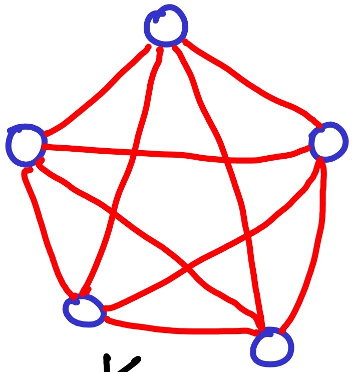
PLANAR GRAPH

can redraw without crossings

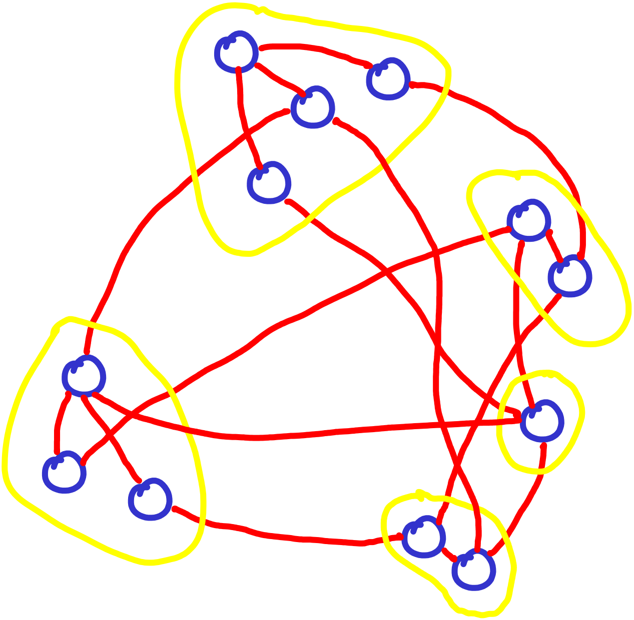
Non-planar graphs (can't redraw)



$K_{3,3}$

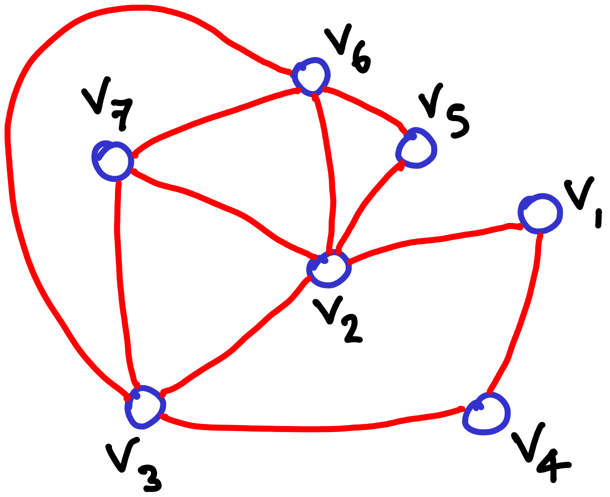


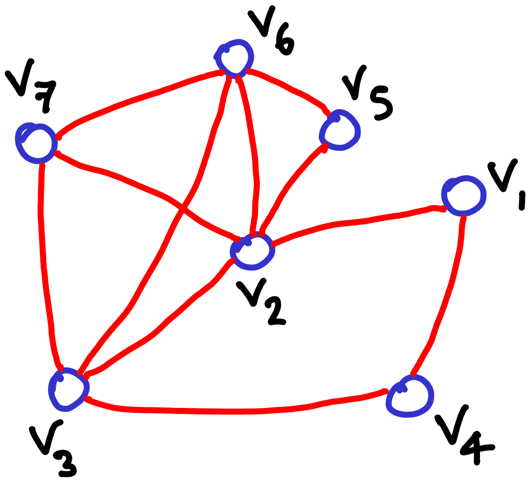
K_5



PLANE GRAPH

no crossings



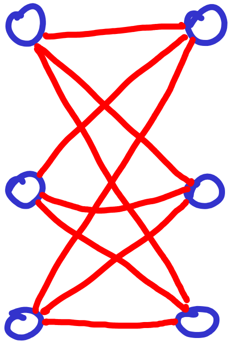


PLANAR GRAPH

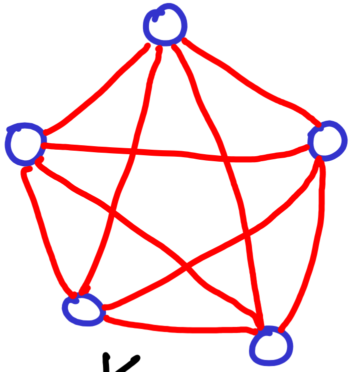
can redraw without crossings

Non-planar graphs

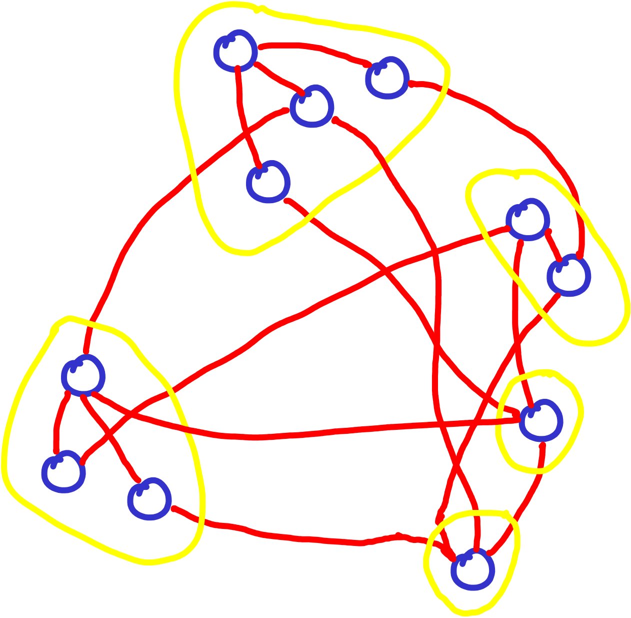
(can't redraw)



$K_{3,3}$

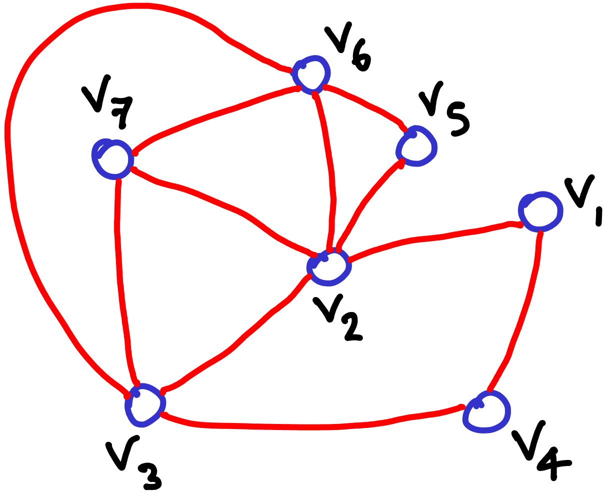


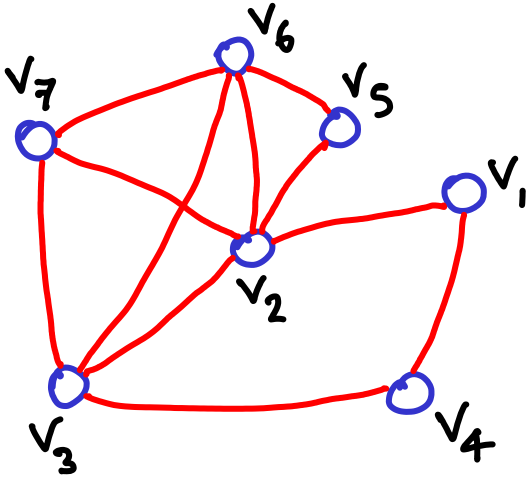
K_5



PLANE GRAPH

no crossings

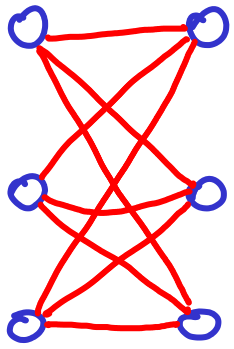




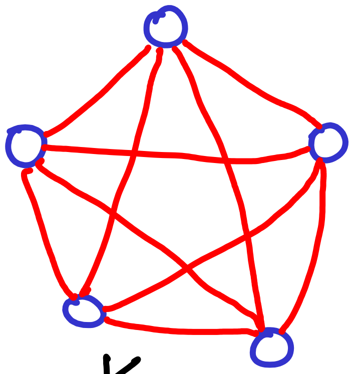
PLANAR GRAPH

can redraw without crossings

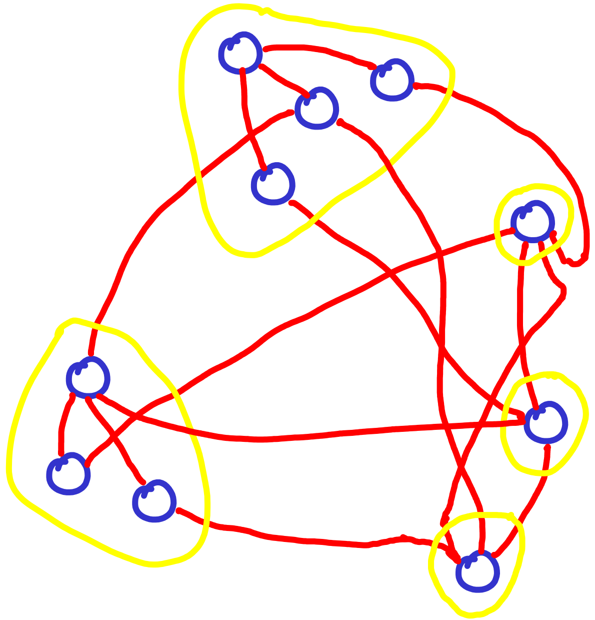
Non-planar graphs (can't redraw)



$K_{3,3}$

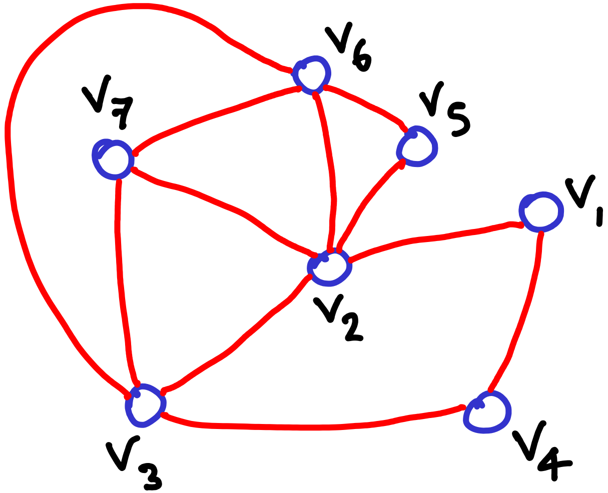


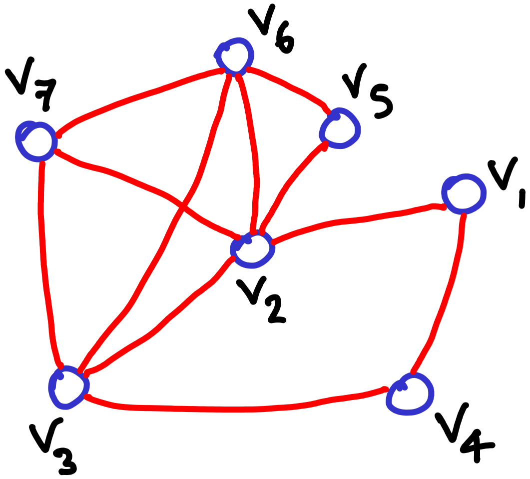
K_5



PLANE GRAPH

no crossings

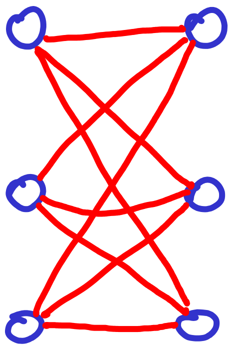




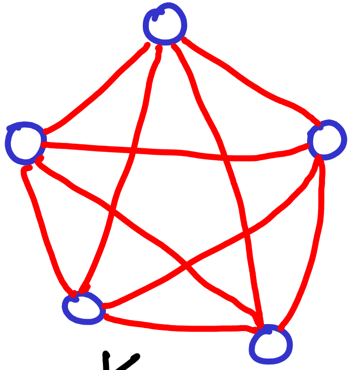
PLANAR GRAPH

can redraw without crossings

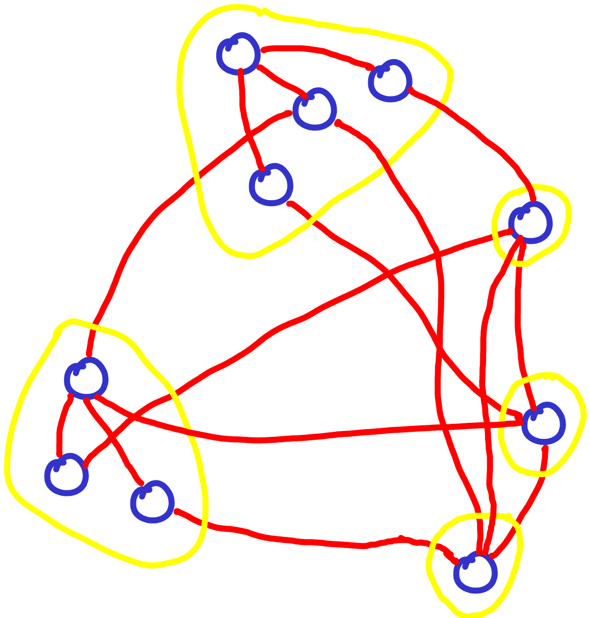
Non-planar graphs (can't redraw)



$K_{3,3}$

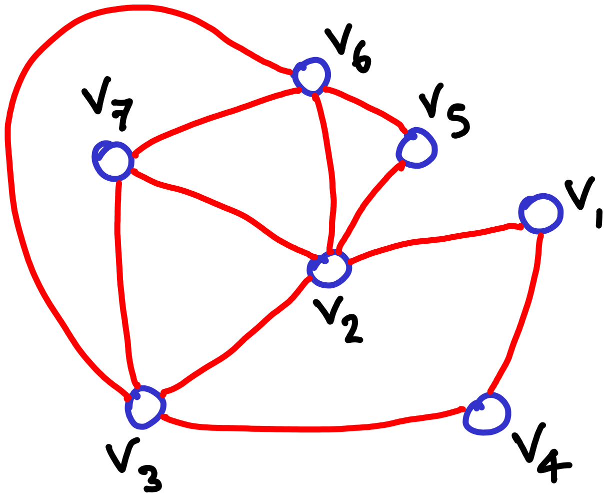


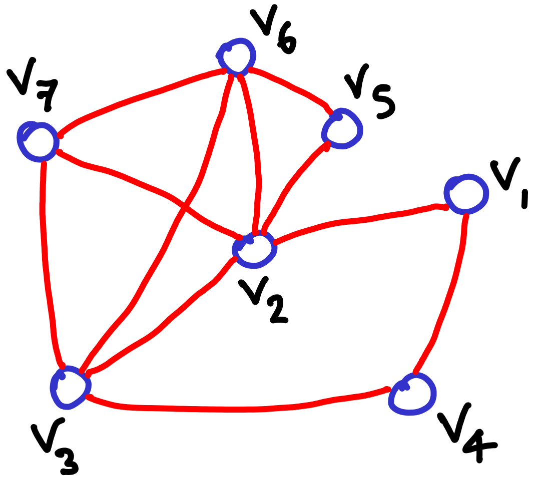
K_5



PLANE GRAPH

no crossings



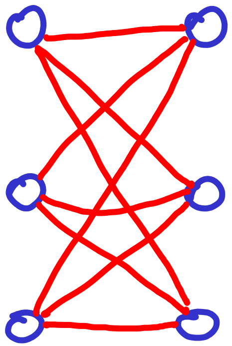


PLANAR GRAPH

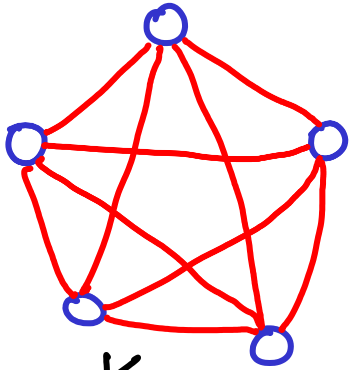
can redraw without crossings

Non-planar graphs

(can't redraw)



$K_{3,3}$



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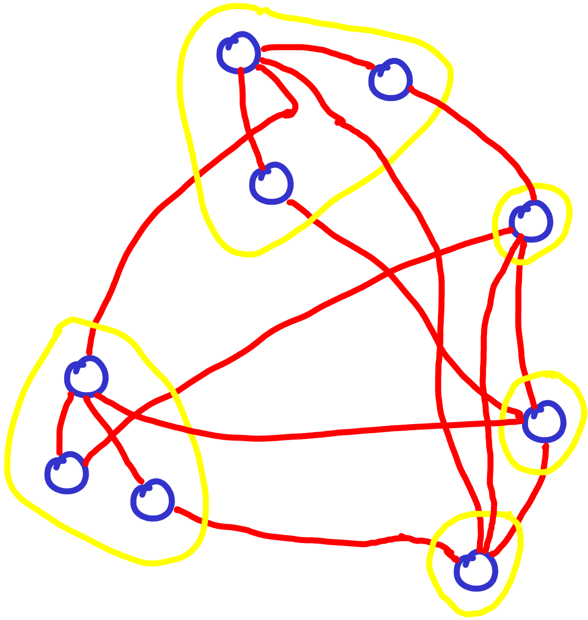
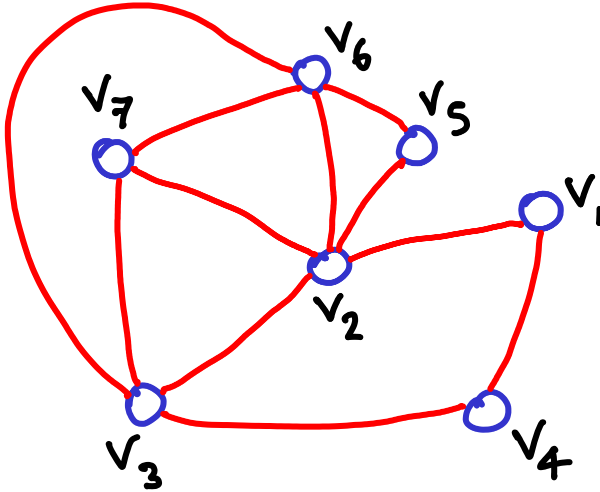
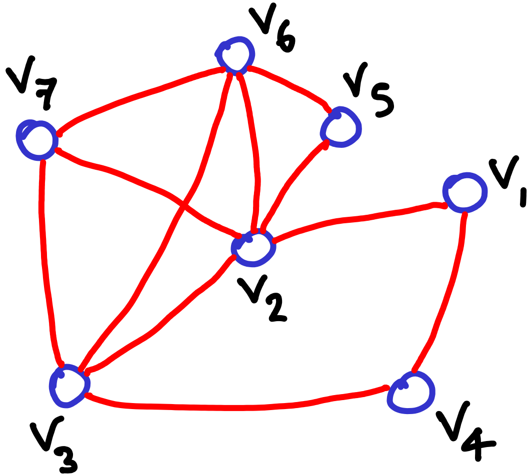


PLANE GRAPH

no crossings



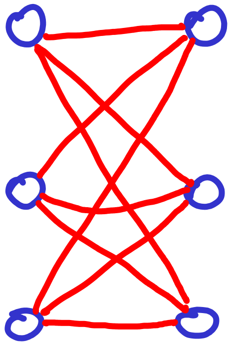


PLANAR GRAPH

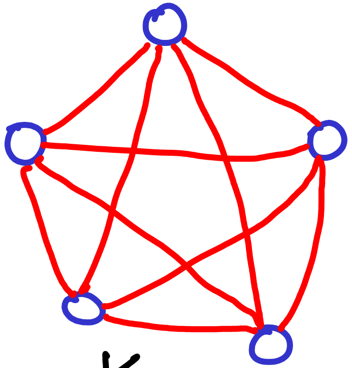
can redraw without crossings

Non-planar graphs

(can't redraw)



$K_{3,3}$



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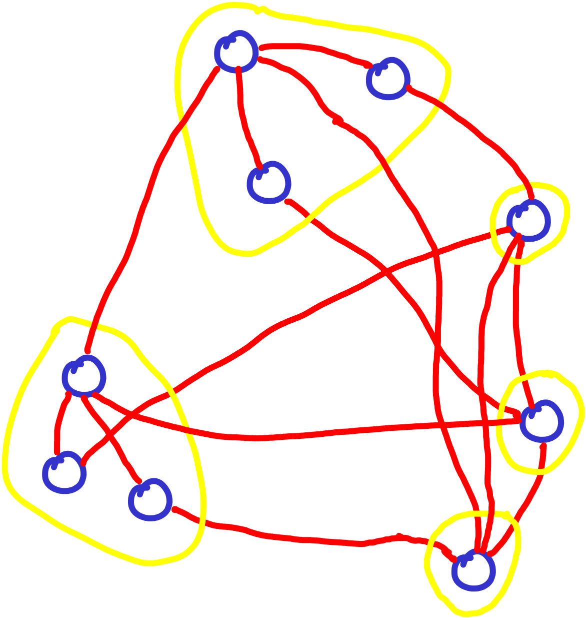
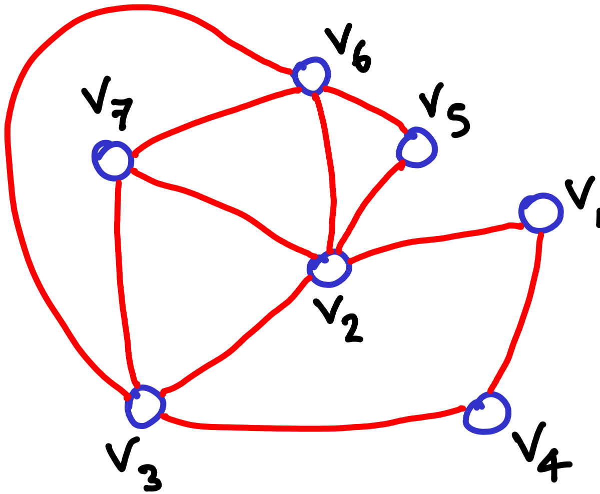
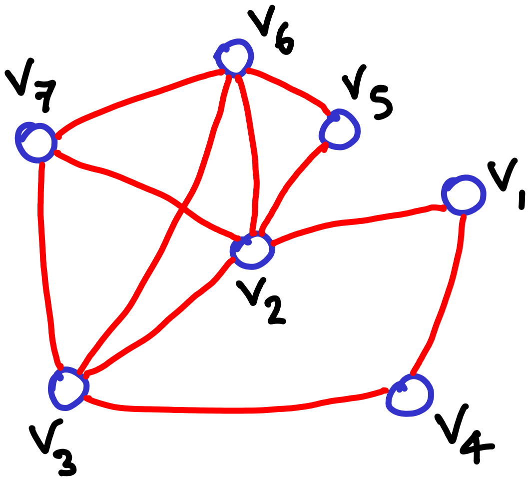


PLANE GRAPH

no crossings

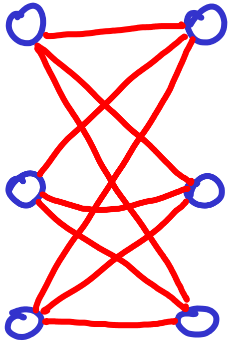




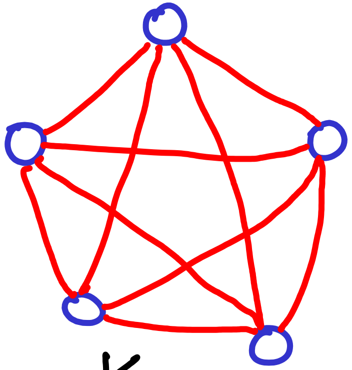
PLANAR GRAPH

can redraw without crossings

Non-planar graphs (can't redraw)



$K_{3,3}$



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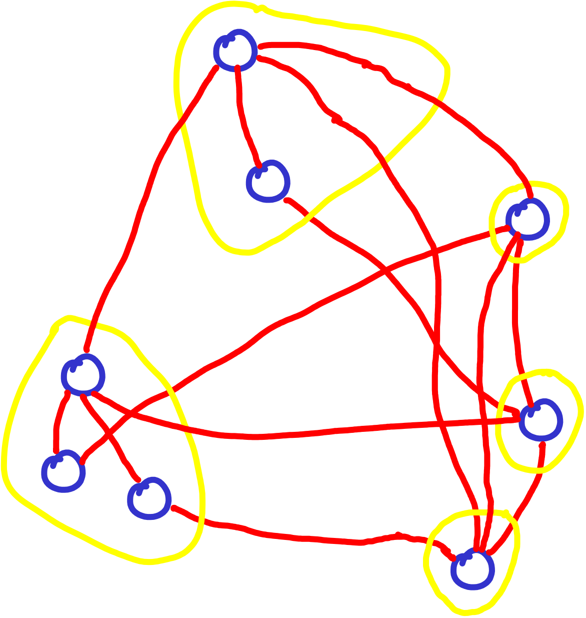
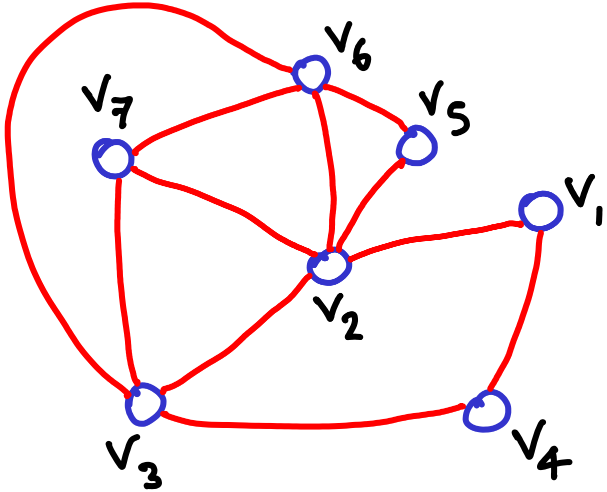
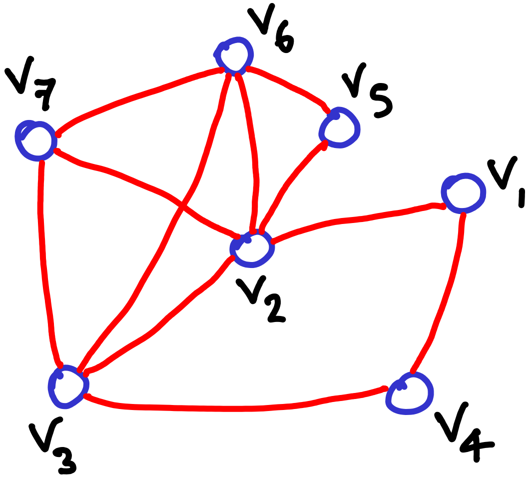


PLANE GRAPH

no crossings

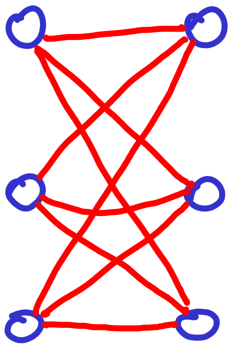




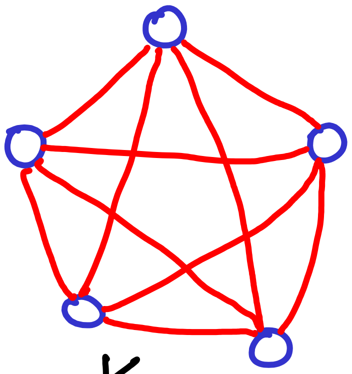
PLANAR GRAPH

can redraw without crossings

Non-planar graphs
(can't redraw)



$K_{3,3}$



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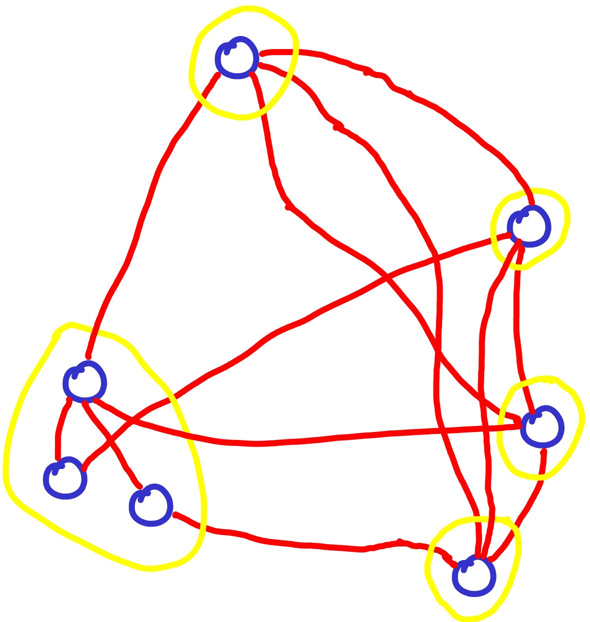
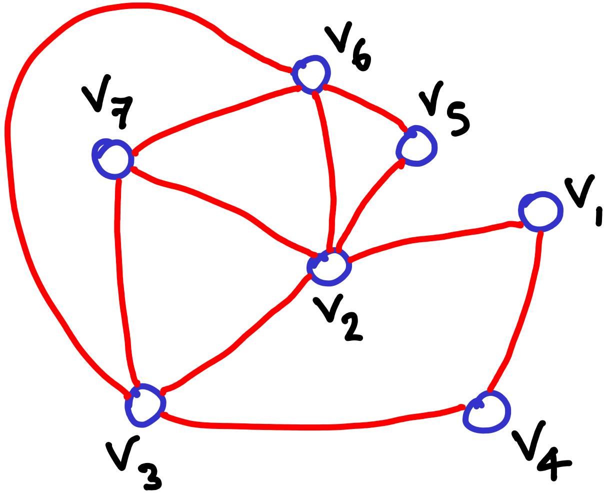
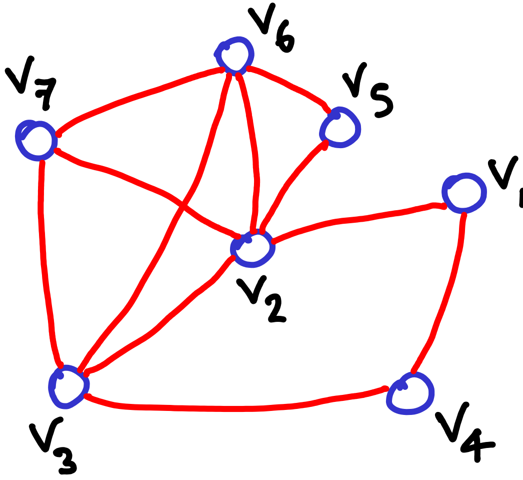


PLANE GRAPH

no crossings

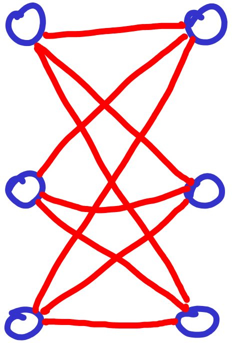




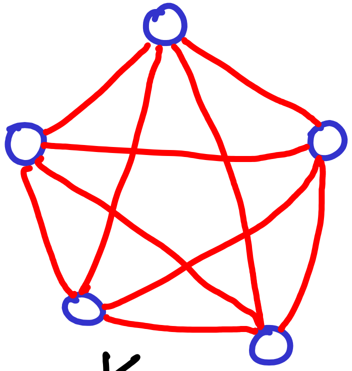
PLANAR GRAPH

can redraw without crossings

Non-planar graphs (can't redraw)



$K_{3,3}$



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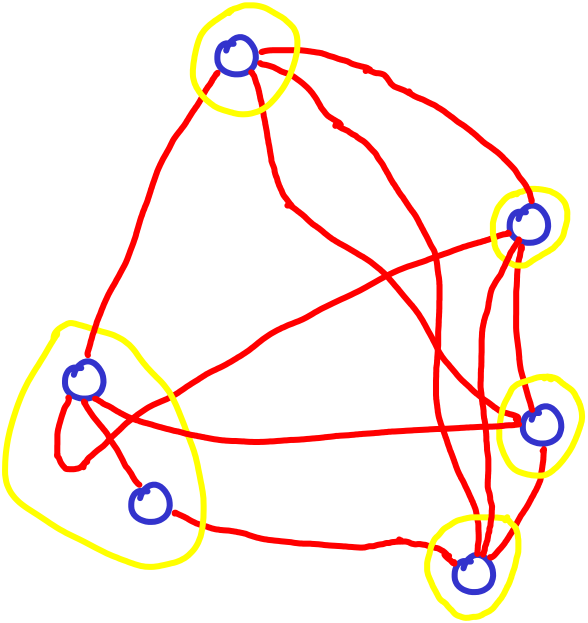
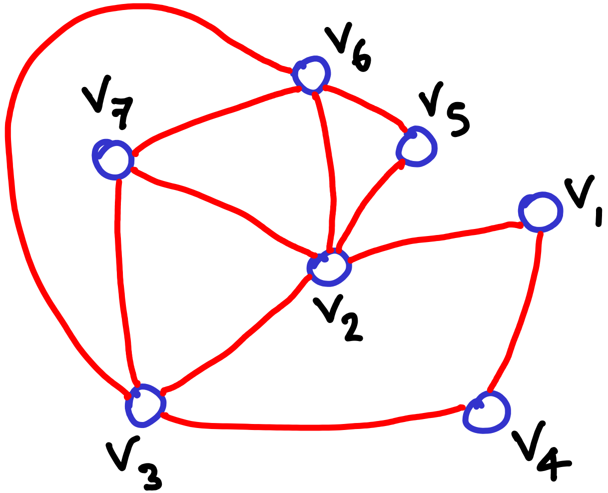
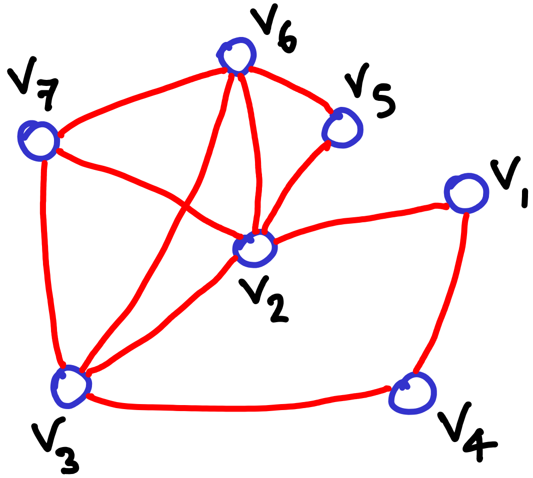


PLANE GRAPH

no crossings

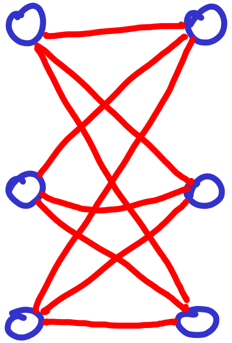




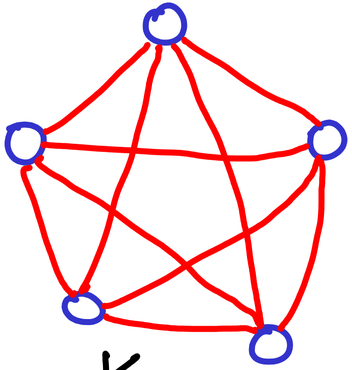
PLANAR GRAPH

can redraw without crossings

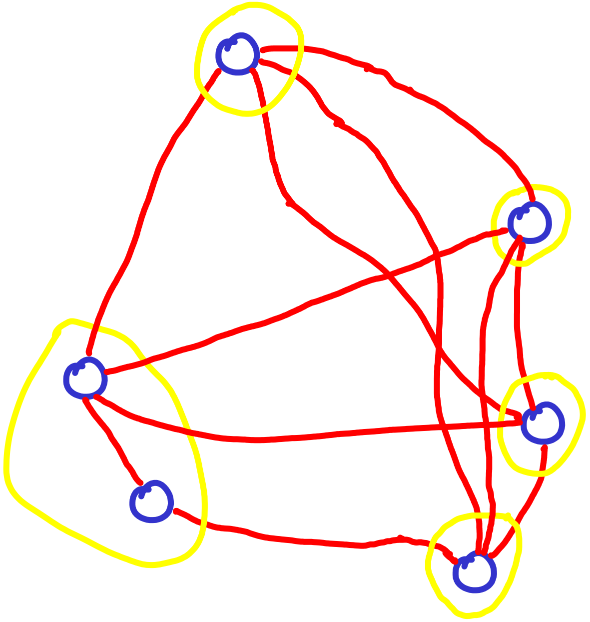
Non-planar graphs (can't redraw)



$K_{3,3}$

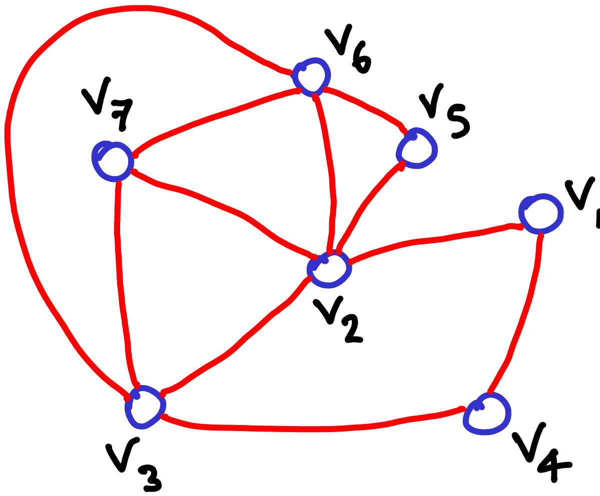


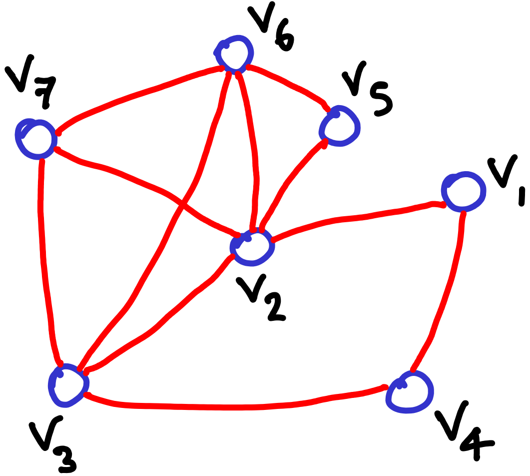
K_5



PLANE GRAPH

no crossings

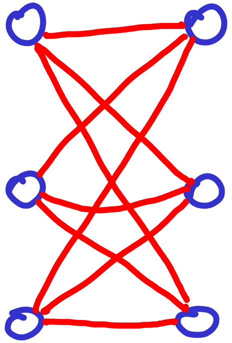




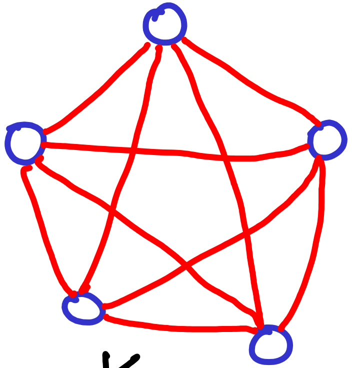
PLANAR GRAPH

can redraw without crossings

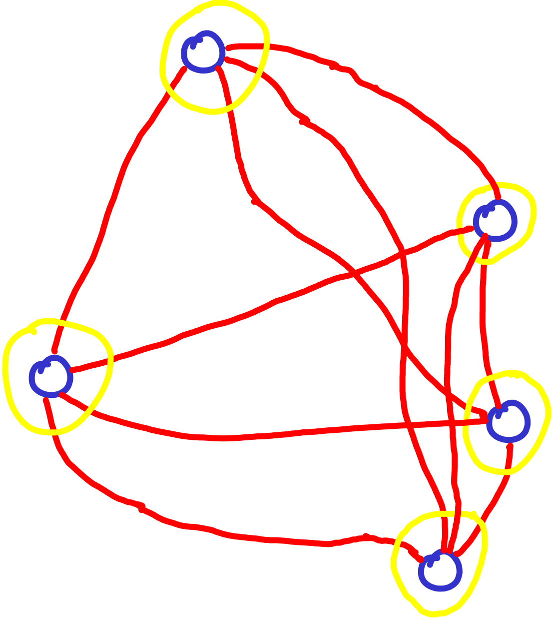
Non-planar graphs (can't redraw)



$K_{3,3}$

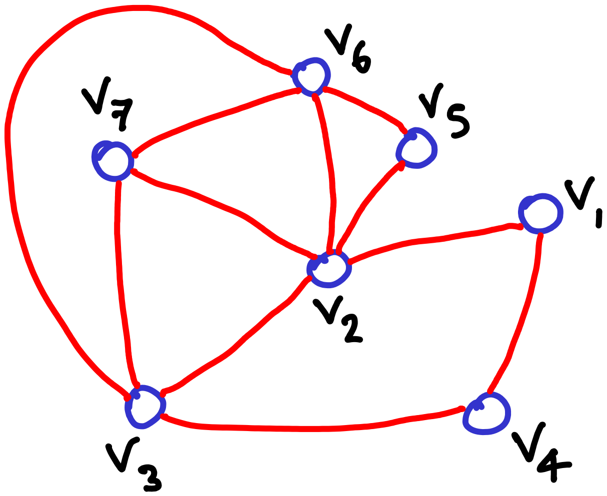


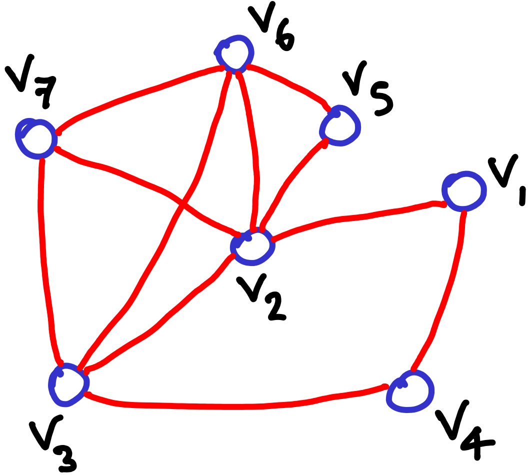
K_5



PLANE GRAPH

no crossings

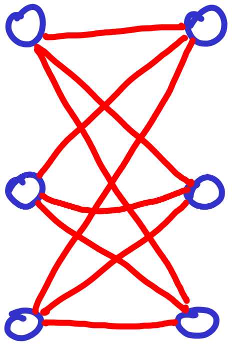




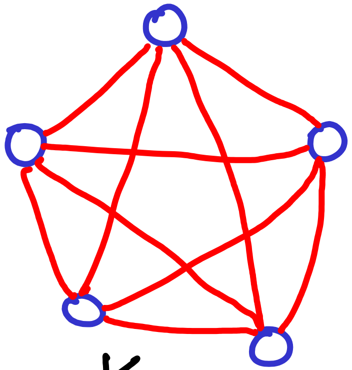
PLANAR GRAPH

can redraw without crossings

Non-planar graphs (can't redraw)

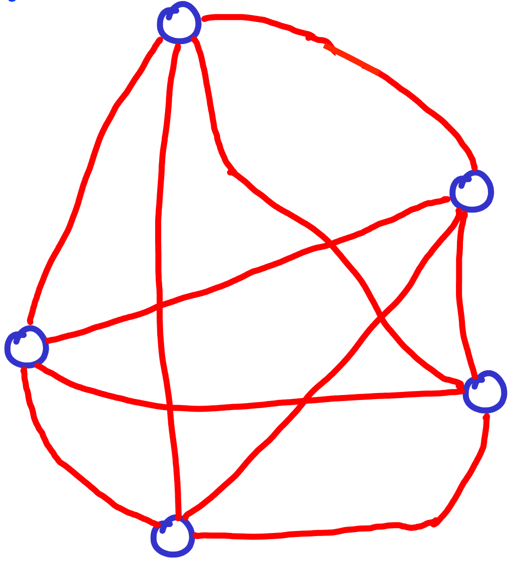


$K_{3,3}$



K_5

obtained by successive contractions



A graph is non-planar if and only if it "contains" a K3,3 or K5

FYI

Every planar graph can be drawn without crossings.
In fact the edges can be drawn straight as well.

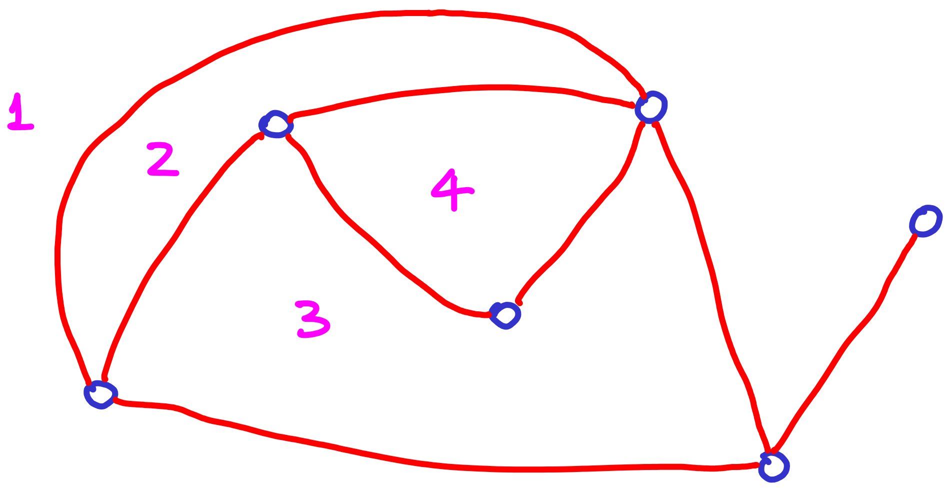
$G=(V,E)$

$V=6$

$E=8$

$F = \#\text{faces} = 4$

disjoint regions, one of which is unbounded



EULER FORMULA for planar connected graphs: $\underline{V - E + F = 2}$

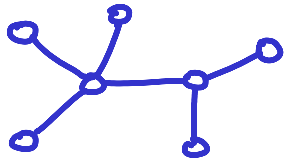
EULER FORMULA for planar connected graphs: $V - E + F = 2$

Proof by induction on number of faces:

Base case → $F = 1$ → ?

EULER FORMULA for planar connected graphs: $V - E + F = 2$

Proof by induction on number of faces:



Base case → $F = 1$ → G is a tree → ?

EULER FORMULA for planar connected graphs: $V - E + F = 2$

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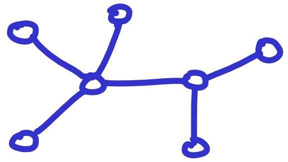

Base case $\rightarrow F = 1 \rightarrow$ G is a tree $\rightarrow V = E + 1$

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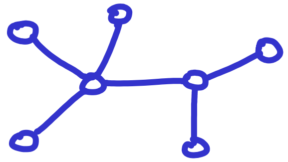

Base case → $F = 1$ → G is a tree → $V = E + 1$

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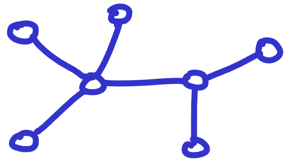

Base case → $F = 1$ → G is a tree → $V = E + 1$

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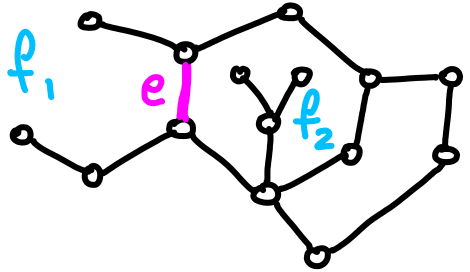
Given $G = (V, E)$ w/ $F > 1$ faces,



remove an edge e between 2 faces, f_1 & f_2.

EULER FORMULA for planar connected graphs: $V - E + F = 2$

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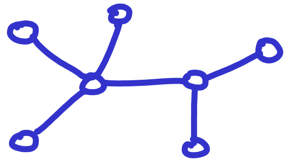

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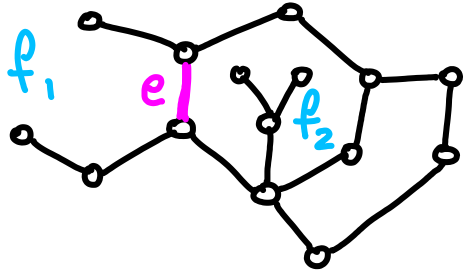

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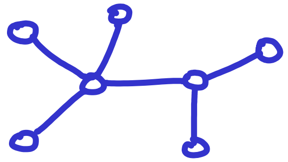

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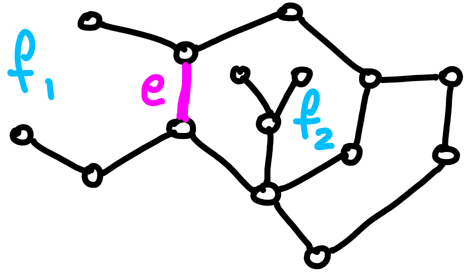

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Either f_1 or f_2 is a bounded face,

so e is on a cycle (e is not a cut edge)

EULER FORMULA for planar connected graphs: $V - E + F = 2$

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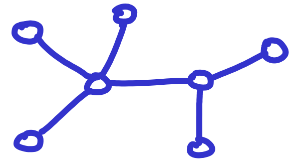

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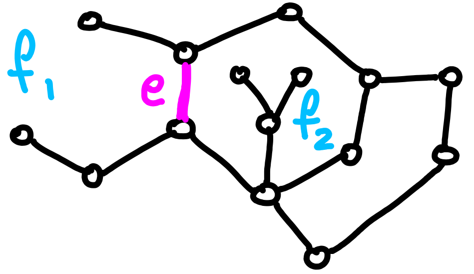

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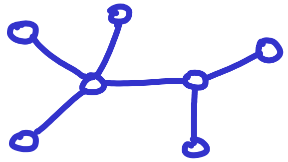

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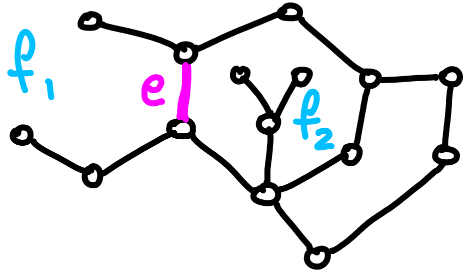

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$$\boxed{\underbrace{V \quad (E-1) \quad (F-1)}_{G-e}}$$

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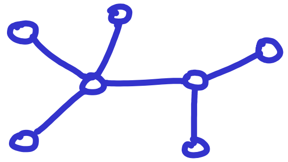

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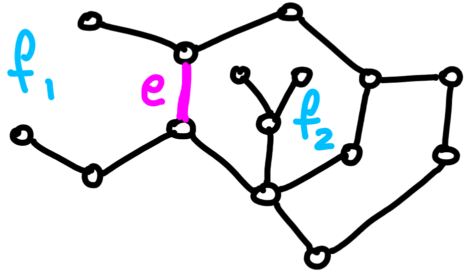

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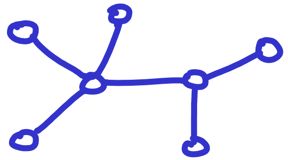
EULER FORMULA for planar connected graphs: $\underline{V - E + F = 2}$

Proof by induction on number of faces:



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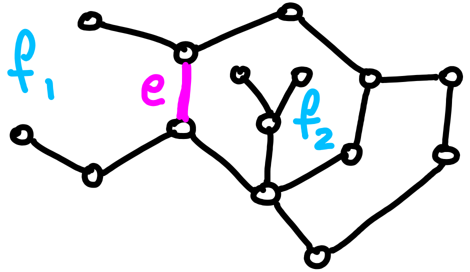

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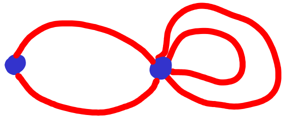
$\hookrightarrow V - E + F = 2$ ✓

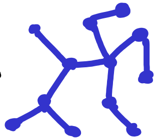
Note that this also holds for multigraphs



Euler formula $V - E + F = 2$

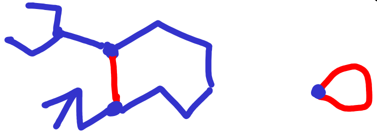
$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F = 1$:  : tree. $E = V - 1$

$F > 1$:

Remove an edge between 2 faces.



Remains connected.

$F \to F-1 \quad E \to E-1$

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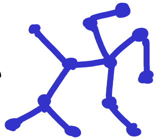
Induction on faces:

$F = 1$:  : tree. $E = V - 1$

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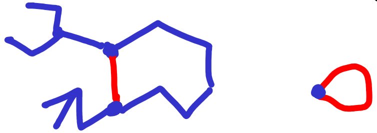

Remains connected.

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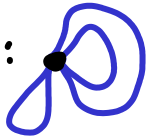
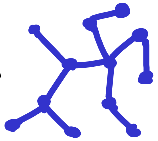
Induction on vertices

$V = 1$:  : only loops $F = E + 1$

Euler formula $V - E + F = 2$

$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F = 1$:  : tree. $E = V - 1$

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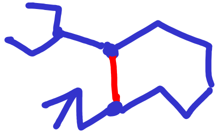
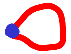
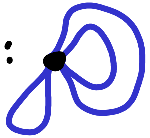




Remains connected.

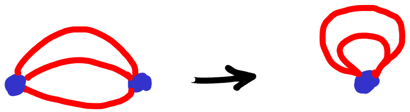
$F \to F-1 \quad E \to E-1$

Induction on vertices

$V = 1$:  : only loops
$F = E + 1$

$V > 1$:
Contract edge $x \neq y$ 

$V \to V-1 \quad E \to E-1$



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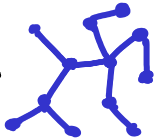
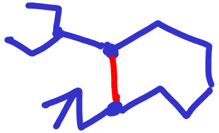
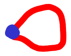
Induction on faces:

$F = 1$:  : tree. $E = V - 1$

$F > 1$:

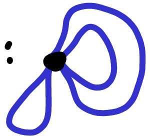
Remove an edge between 2 faces.

Remains connected.

$F \to F-1 \qquad E \to E-1$

Induction on vertices

$V = 1$:  : only loops $F = E + 1$

$V > 1$:

Contract edge $x \neq y$ 

$V \to V-1 \qquad E \to E-1$

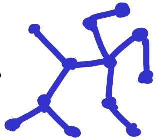
Induction on edges

$E = 0$:  : one vertex one face

Euler formula $V-E+F=2$

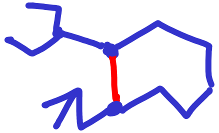
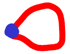
$V-E+F=2$ applies to any connected planar graph (in fact, to convex polyhedra by projection)

Induction on faces:

$F=1$:  : tree. $E=V-1$

$F>1$:

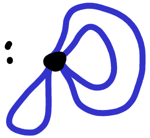
Remove an edge between 2 faces.

Remains connected.

$F \to F-1 \qquad E \to E-1$

Induction on vertices

$V=1$:  : only loops $F=E+1$

$V>1$:

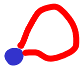
Contract edge $x \neq y$ 

$V \to V-1 \qquad E \to E-1$

Induction on edges

$E=0$: • : one vertex one face

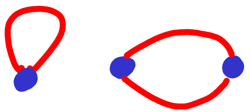
$E \geq 1$: if $x \neq y$  contract as before

else  remove as before

$E \to E-1$ & $F \to F-1$ OR $V \to V-1$

use the Euler formula $V - E + F = 2$

to show that a connected plane graph has $E \leq 3V - 6$ for $V > 3$

Not allowed: 

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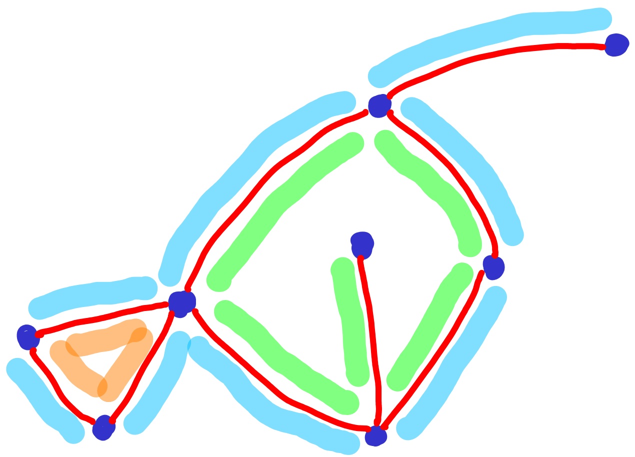
Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$



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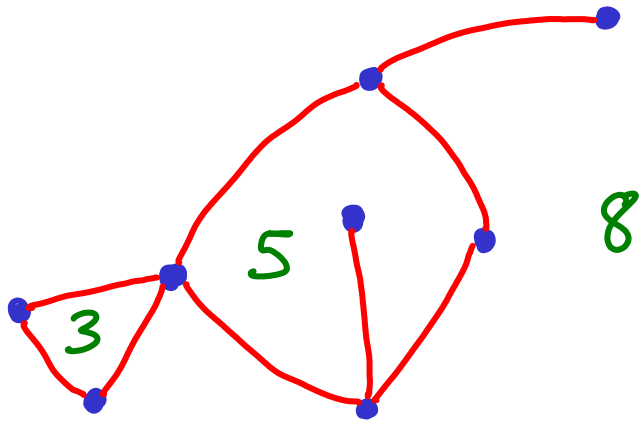
Every edge belongs to 1 or 2 faces $\displaystyle\sum_{\text{all faces}} e \leq 2E$

Every face has $\geqslant 3$ edges (for $V > 3$) $\displaystyle\sum_{\text{all faces}} e \geqslant 3F$



use the Euler formula $V - E + F = 2$

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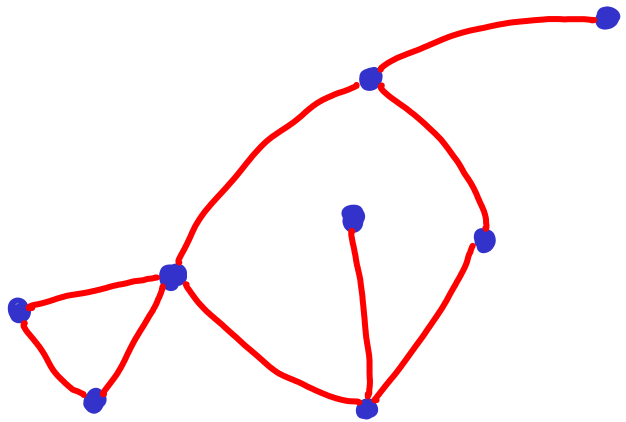
Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V > 3$) $\sum_{\text{all faces}} e \geq 3F$

$$\left.\begin{array}{l} \sum_{\text{all faces}} e \leq 2E \\ \sum_{\text{all faces}} e \geq 3F \end{array}\right\} 2E \geq 3F$$



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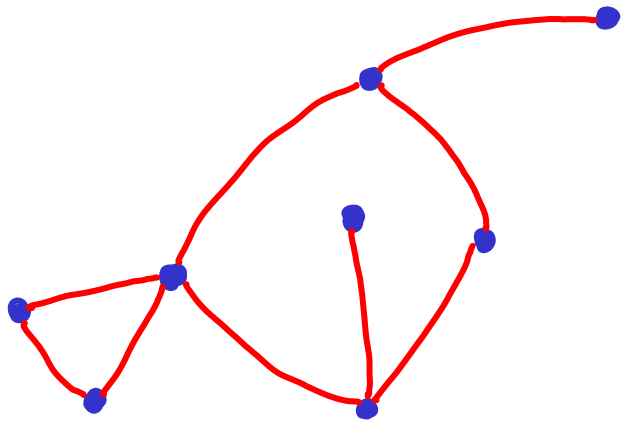
Every edge belongs to 1 or 2 faces $\quad \sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V > 3$) $\quad \sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

$E - F = V - 2$



use the Euler formula $V - E + F = 2$

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Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$

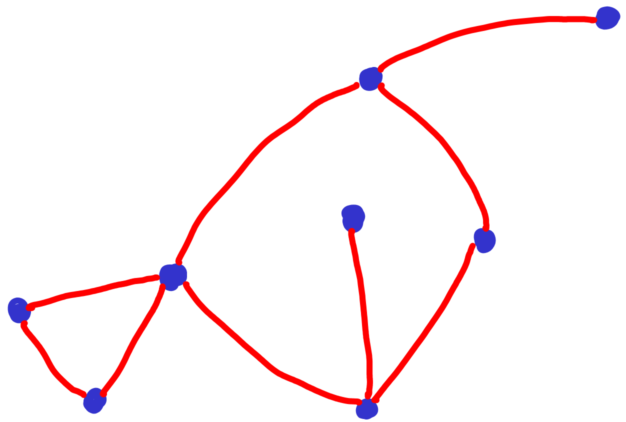
Every face has ≥ 3 edges (for $V > 3$) $\sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

$$E - F = V - 2$$

$$E - \frac{2E}{3} \leq V - 2$$

$$E \leq 3V - 6$$



use the Euler formula $V - E + F = 2$

to show that a connected plane graph has $E \leq 3V - 6$

Every edge belongs to 1 or 2 faces $\qquad \sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V > 3$) $\qquad \sum_{\text{all faces}} e \geq 3F$

$$2E \geq 3F$$

$$E - F = V - 2$$

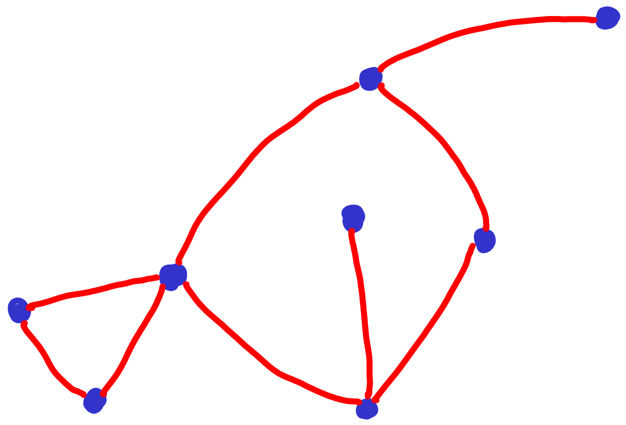
$$E - \frac{2E}{3} \leq V - 2$$

$$E \leq 3V - 6$$

$$E - F = V - 2$$

$$\frac{3F}{2} - F \leq V - 2$$

$$F \leq 2V - 4$$



$E \leq 3V - 6$

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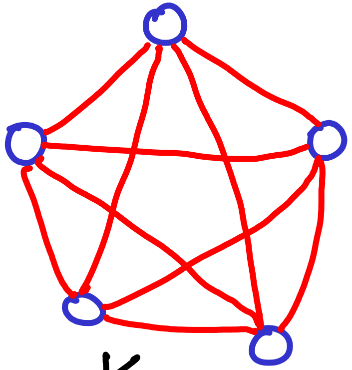
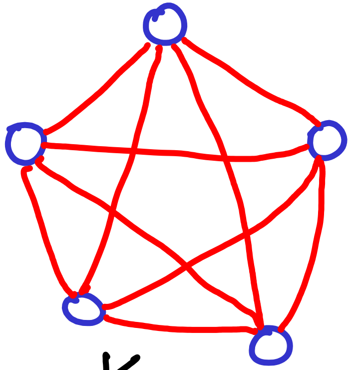


K_5

$E \leq 3V - 6$

$10 \leq 15 - 6$

!!!



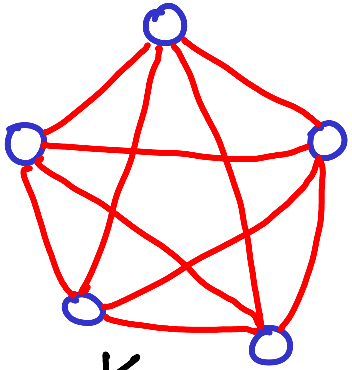
K_5

Not planar

$E \leq 3V - 6$

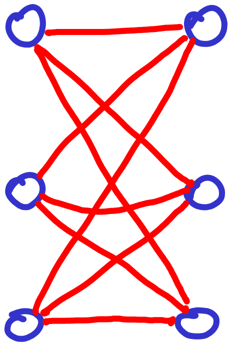
$10 \leq 15 - 6$

!!!



K_5

$E \leq 3V - 6$

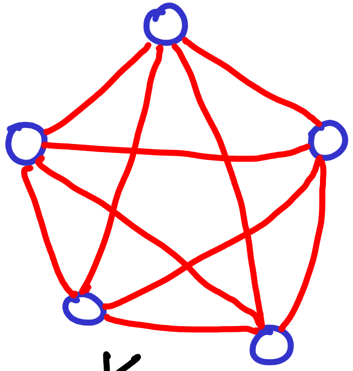


$K_{3,3}$

$E \leq 3V-6$

$10 \leq 15-6$

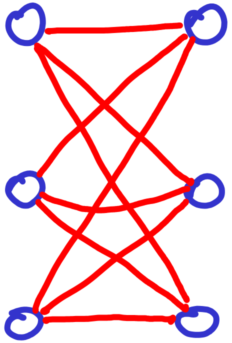
!!!



K_5

$E \leq 3V-6$

$9 \leq 18-6$ ok!

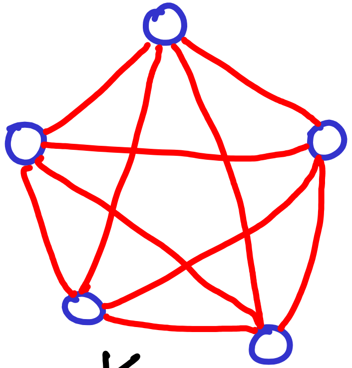


$K_{3,3}$

$E \leq 3V - 6$

$10 \leq 15 - 6$

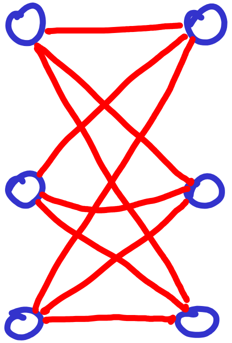
!!!



K_5

$E \leq 3V - 6$

$9 \leq 18 - 6$ ok!



$K_{3,3}$

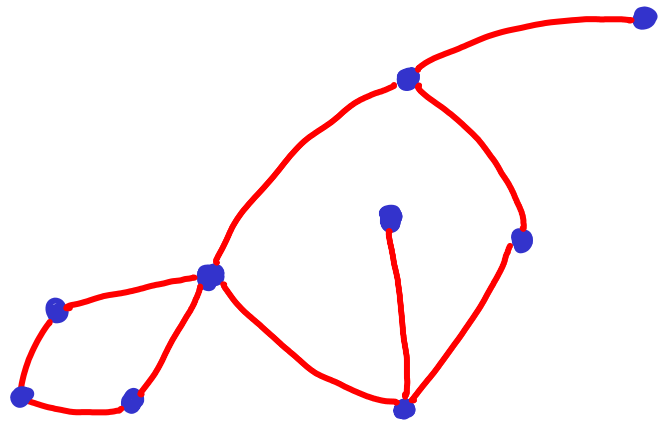
not iff

All planar graphs have $E \leq 3V-6$

Some non-planar graphs can too

$V - E + F = 2$

What if G has no triangles?

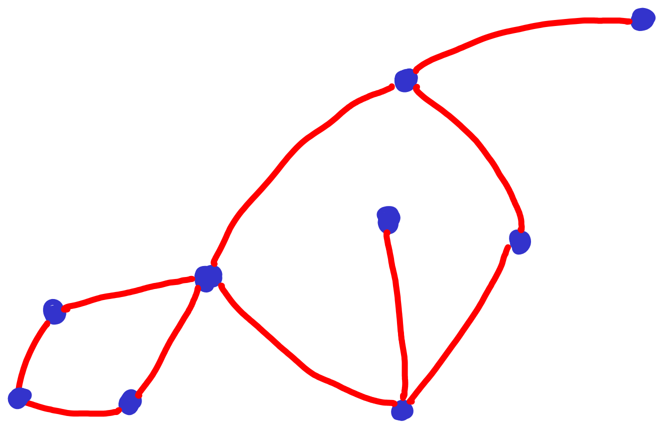


$V - E + F = 2$ What if G has no triangles?

Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$

Every face has $\cancel{\geq 3}$ ≥ 4 edges (for $V>4$) $\sum_{\text{all faces}} e \geq 4F$

$E \geq 2F$



$V - E + F = 2$

What if G has no triangles?

Every edge belongs to 1 or 2 faces $\qquad \sum_{\text{all faces}} e \leq 2E$

Every face has $\cancel{\geq 3}$ ≥ 4 edges (for $V > 4$) $\qquad \sum_{\text{all faces}} e \geq 4F$

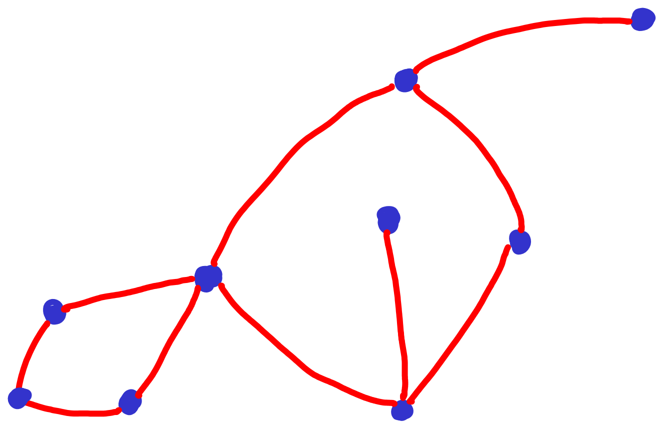
$E \geq 2F$

$E - F = V - 2$

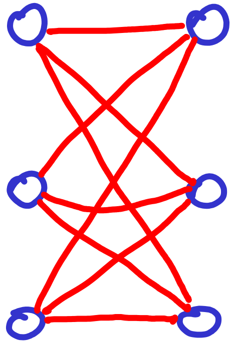
$E - \frac{E}{2} \leq V - 2$

$E \leq 2V - 4$

instead of $\leq 3V - 6$



triangle free



$K_{3,3}$

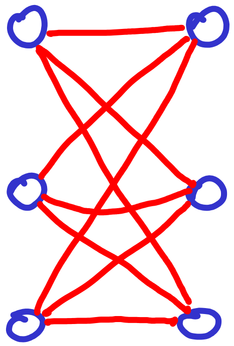
for triangle free: $E \leq 2V - 4$

$9 \leq 2 \cdot 6 - 4$

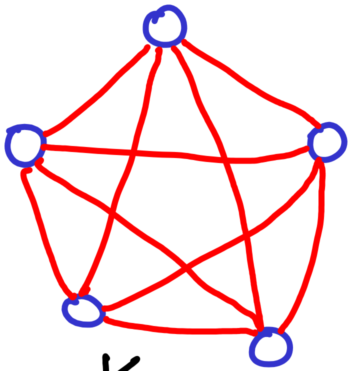
!!!

NOT PLANAR

$V = 6, \ E = 9$

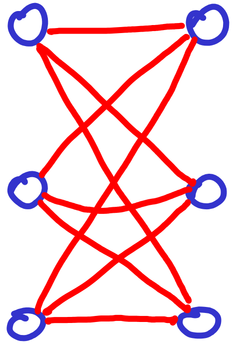


$K_{3,3}$



K_5

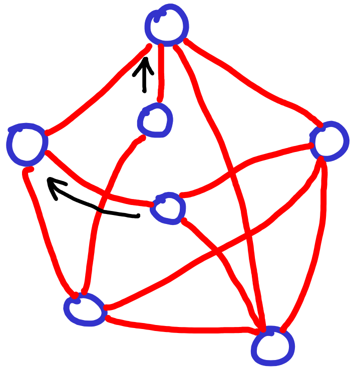
← non-planar →



$K_{3,3}$

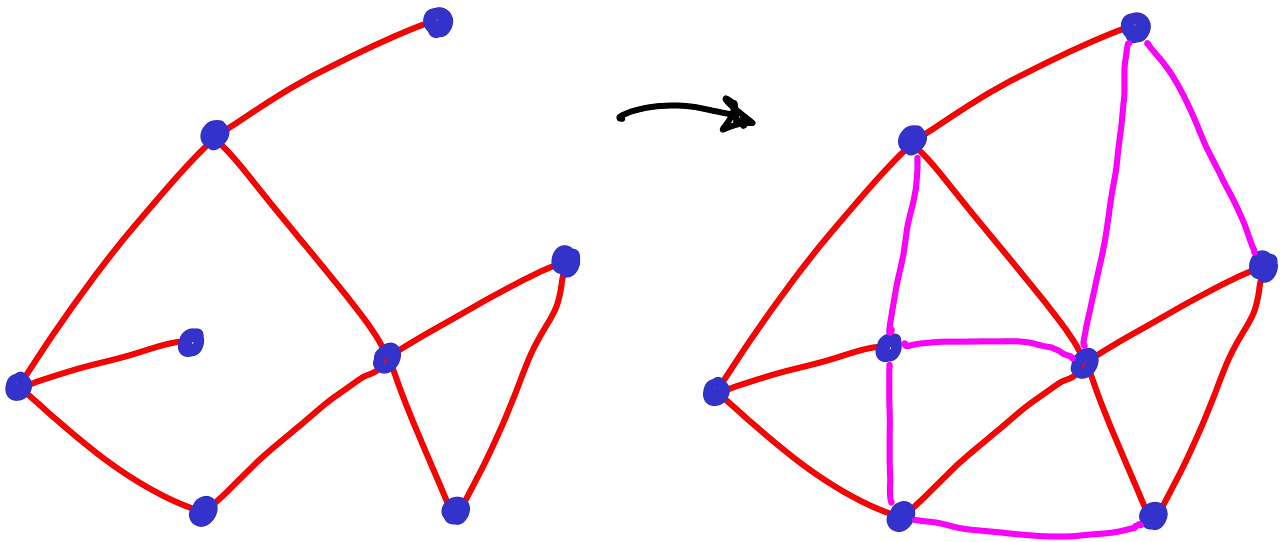
It turns out that every non-planar graph "contains" one of these two shapes.

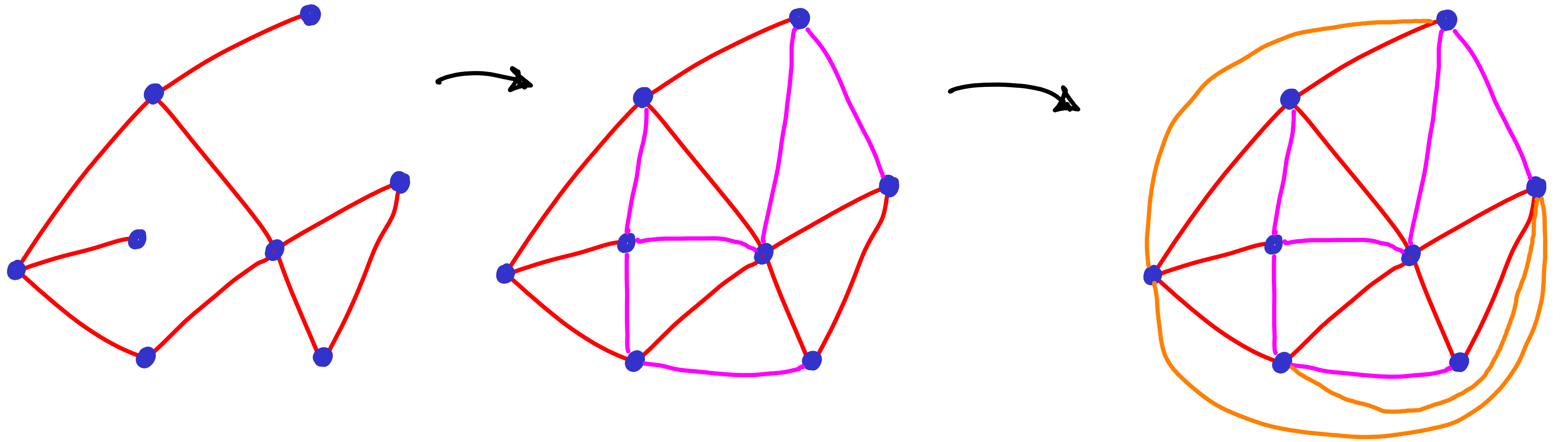
e.g:



see links

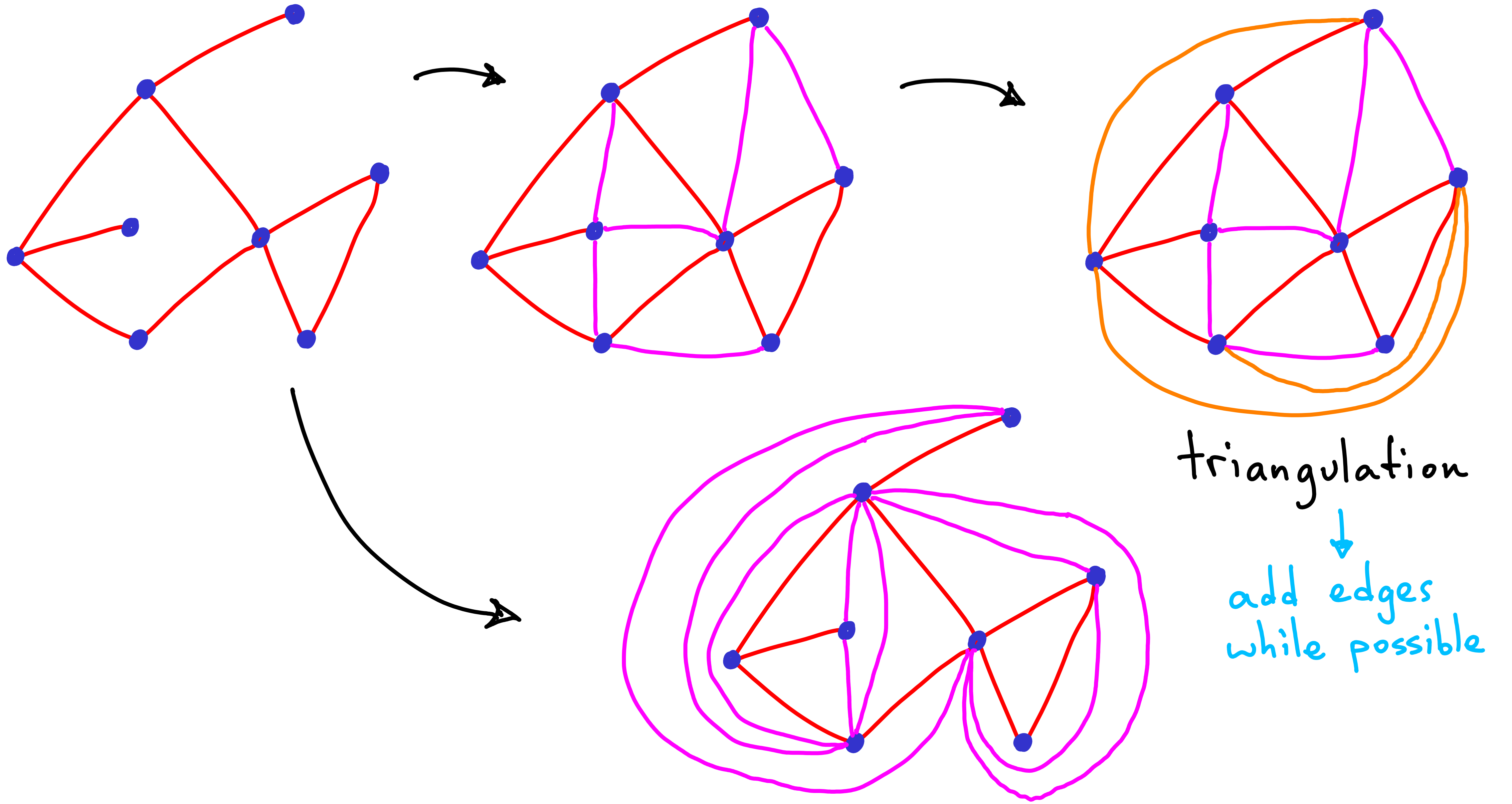
TRIANGULATIONS





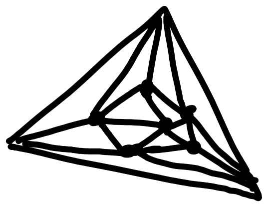
triangulation

add edges while possible


triangulation
add edges
while possible

$E = 3V - 6$

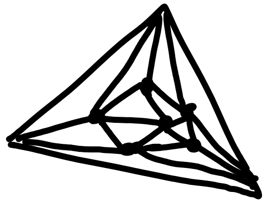
Why?



Assume outer face is a triangle

$\underline{\underline{E = 3V - 6}}$

Why?



$\Big\{$ Assume outer face is a triangle

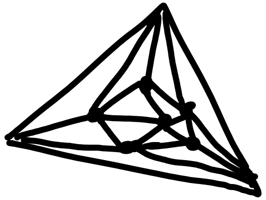
Every edge belongs to 1 or 2 faces $\quad \sum_{\text{all faces}} e \leq 2E$

Every face has ≥ 3 edges (for $V > 3$) $\quad \sum_{\text{all faces}} e \geq 3F$

$2E \geq 3F$

$E = 3V - 6$

Why?



Assume outer face is a triangle

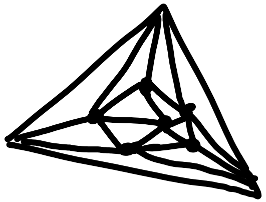
Every edge belongs to ~~1 or~~ 2 faces $\quad \sum_{\text{all faces}} e \overset{=}{\cancel{\leq}} 2E$

Every face has $\overset{=}{\cancel{\geq}}$ 3 edges (for $V>3$) $\quad \sum_{\text{all faces}} e \overset{=}{\cancel{\geq}} 3F$

$2E \overset{=}{\cancel{\geq}} 3F$

$\underline{\underline{E = 3V - 6}}$

Why?



$\left\{ \text{Assume outer face is a triangle} \right.$

$\boxed{V - E + F = 2}$

Every edge belongs to ~~1 or~~ 2 faces $\qquad \displaystyle\sum_{\text{all faces}} e \overset{=}{\cancel{\leq}} 2E$

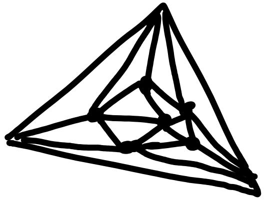
Every face has $\overset{=}{\cancel{\geq}}$ 3 edges (for $V > 3$) $\qquad \displaystyle\sum_{\text{all faces}} e \overset{=}{\cancel{\geq}} 3F$

$2E \overset{=}{\cancel{\geq}} 3F$

$$\boxed{E - F = V - 2}$$

$E = 3V-6$

Why?



Assume outer face is a triangle

$\boxed{V - E + F = 2}$

Every edge belongs to ~~1 or~~ 2 faces

Every face has $\overset{=}{\cancel{\geq}}$ 3 edges (for $V>3$)

$$\sum_{\text{all faces}} e \;\overset{=}{\cancel{\leq}}\; 2E$$

$$\sum_{\text{all faces}} e \;\overset{=}{\cancel{\geq}}\; 3F$$

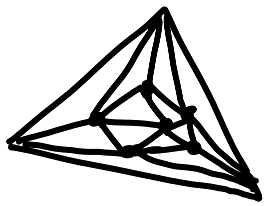
$$2E \;\overset{=}{\cancel{\geq}}\; 3F$$

$$E - F = V - 2$$

$$E - \frac{2E}{3} = V - 2$$

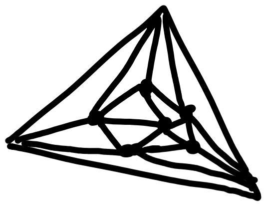
$$E = 3V - 6$$

$E = 3V - 6$



What is the average degree of a triangulation?

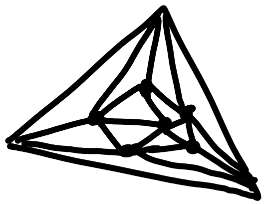
$E = 3V - 6$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i)$$

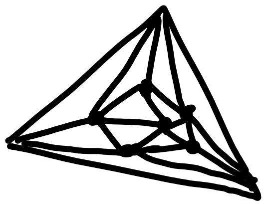
$E = 3V - 6$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E$$

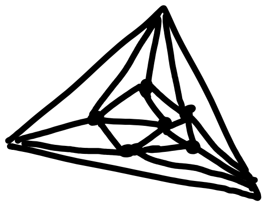
$E = 3V - 6$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

$E = 3V - 6$

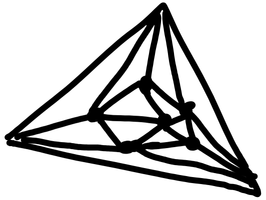


What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

Every triangulation has a vertex w/ degree ≤ 5

$E = 3V - 6$



What is the average degree of a triangulation?

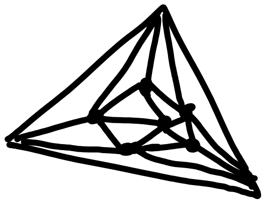
$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

↳ Every triangulation has a vertex w/ degree ≤ 5

↳ Immediately applies to any planar graph

(fewer edges)

$E = 3V - 6$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

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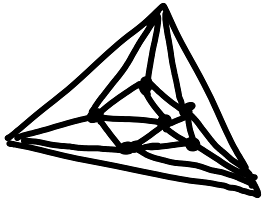
$E = 3V - 6$



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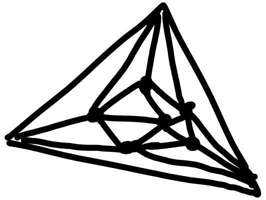
$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices?

$E = 3V - 6$



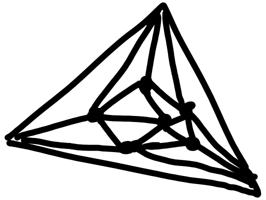
What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 8 ?

$E = 3V - 6$



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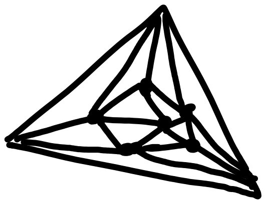
↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 8 ?

↳ $\frac{V}{2}$ degree-8

Prove by contradiction

$E = 3V - 6$



What is the average degree of a triangulation?

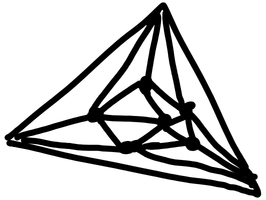
$$\frac{1}{V} \cdot \sum_{i=1}^{V} d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < 6$$

↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find <u>many</u> low-degree vertices? → not if "low" = 5.
what if "low" = 8 ?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

$E = 3V - 6$



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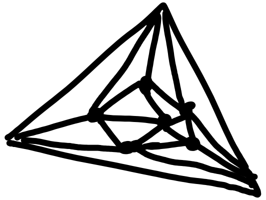
Can we find <u>many</u> low-degree vertices? → not if "low" = 5.
what if "low" = 8 ?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

$$\hookrightarrow \sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$$

sum degrees of $V/2$ of them

$E = 3V - 6$



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↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 8 ?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

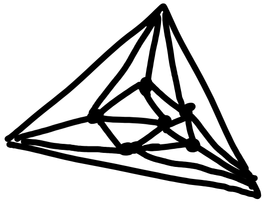
↳ $\sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$

sum degrees of $V/2$ of them

all other vertices have degree ≥ 3

↳ $\sum d(v_i) \geq 3 \cdot \frac{V}{2}$

$E = 3V - 6$



What is the average degree of a triangulation?

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Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 8 ?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

↳ $\sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$

sum degrees of $V/2$ of them

all other vertices have degree ≥ 3

↳ $\sum d(v_i) \geq 3 \cdot \frac{V}{2}$

$\Sigma + \Sigma \geq 6V$

contradiction

$\Downarrow$

always have $\geq \frac{V}{2}$ w/ deg ≤ 8