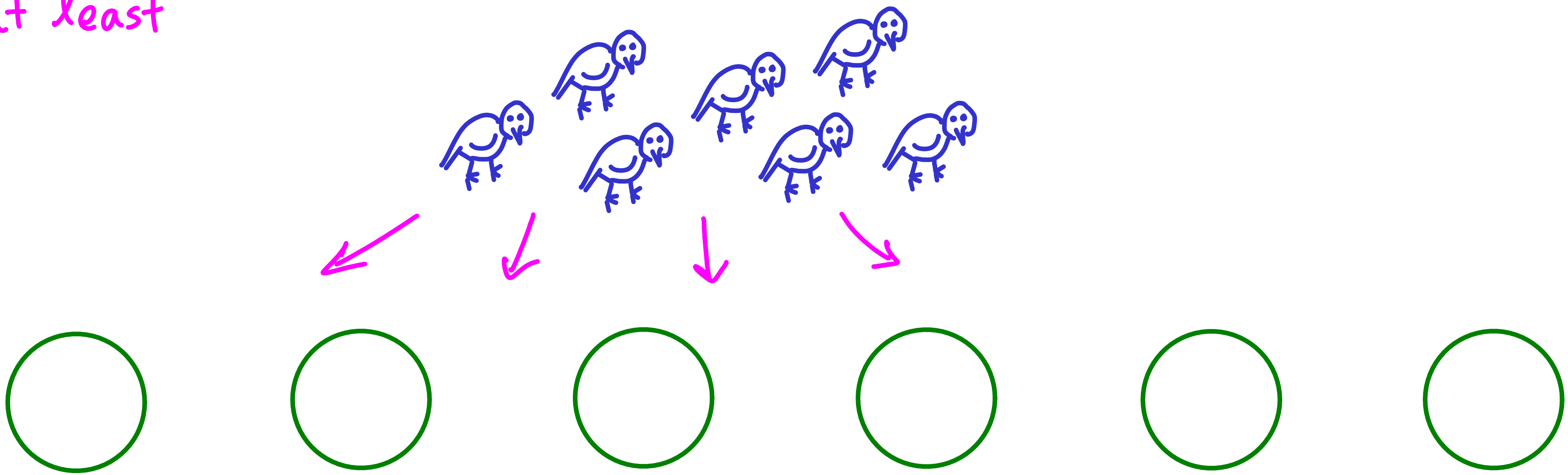
Concepts used in this document

- ceiling function (round up) e.g., $\lceil 1.5 \rceil = 2$
- Set, Subset
- # of subsets that can be formed from a set of size n $= 2^n$
- $\sum_{i=0}^{n} 2^i = 2^n - 1$
- Contrapositive, proof by contradiction

THE PIGEONHOLE PRINCIPLE

If n holes are occupied by $n+1$ pigeons, then $\underbrace{\text{one}}_{\text{at least}}$ hole is occupied by at least two pigeons.



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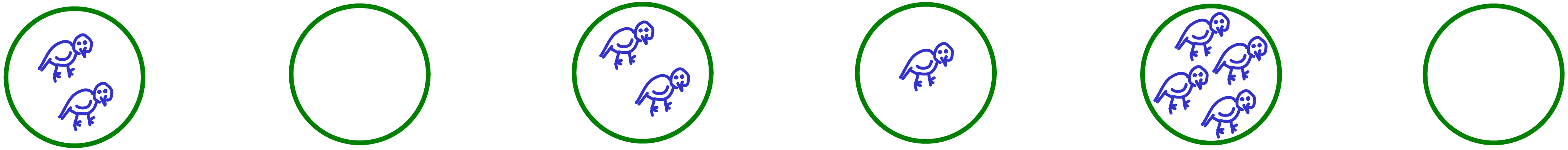
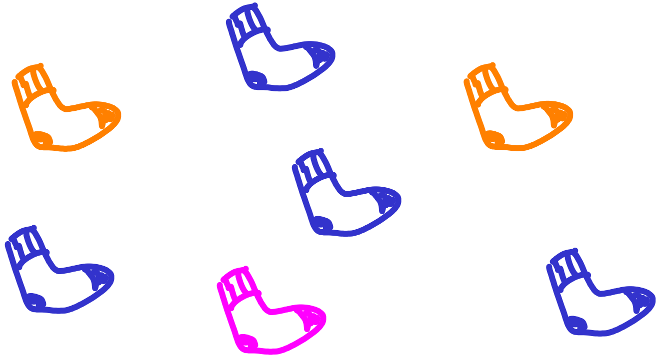


THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of 3 colors,
how many do you need to pick (randomly) to get a matching pair?
(guaranteed)



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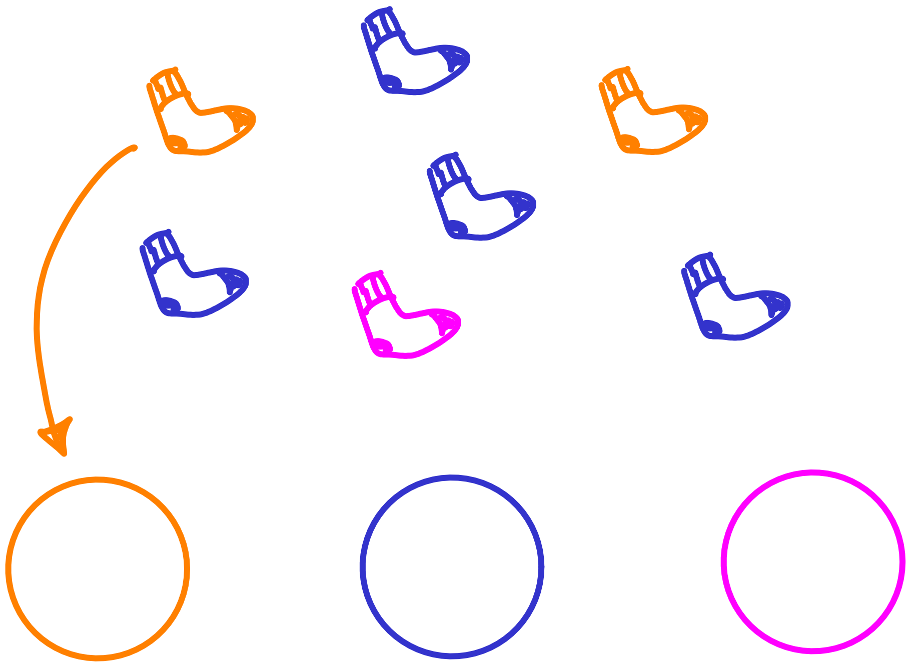

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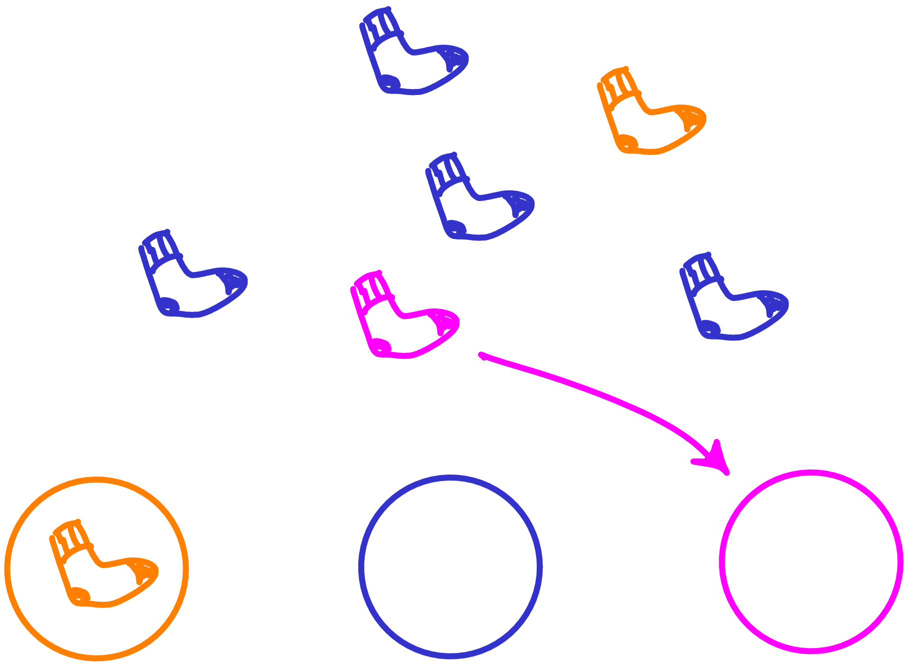

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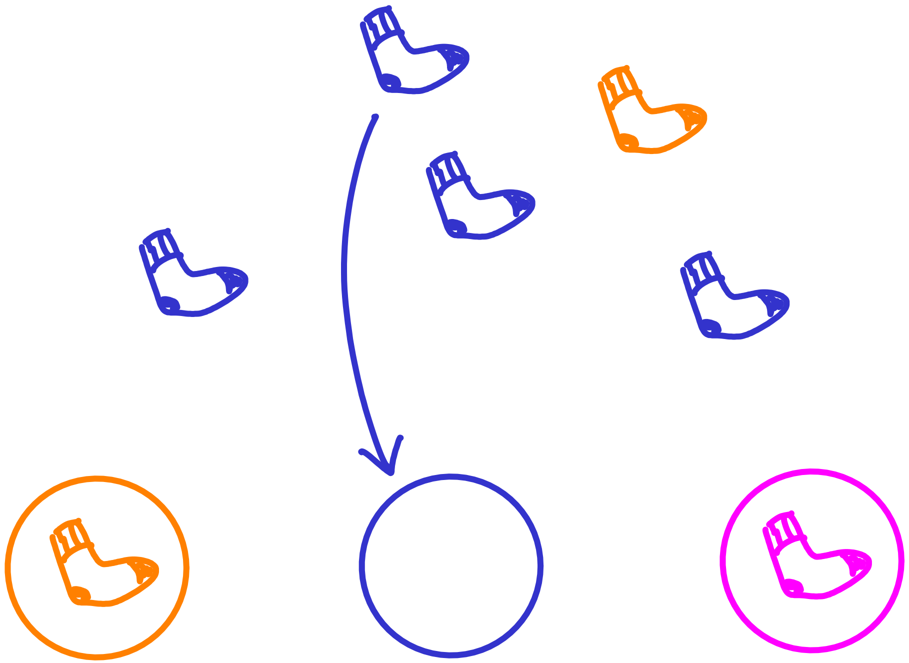

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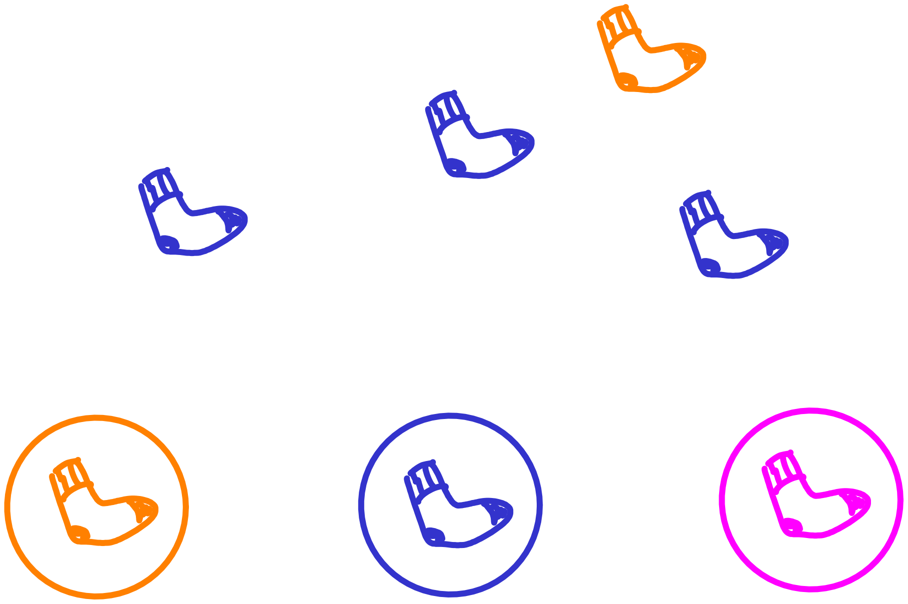
$\boxed{4}$



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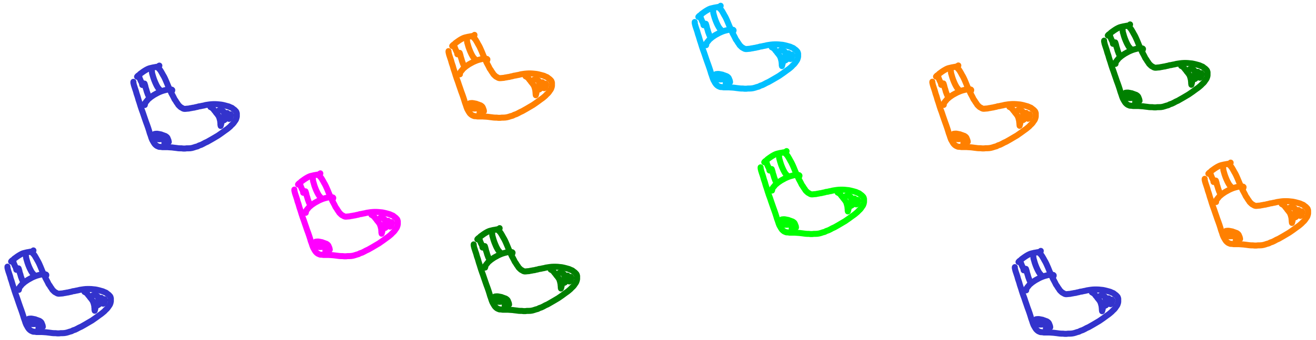

$\boxed{n+1}$

THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of n colors,
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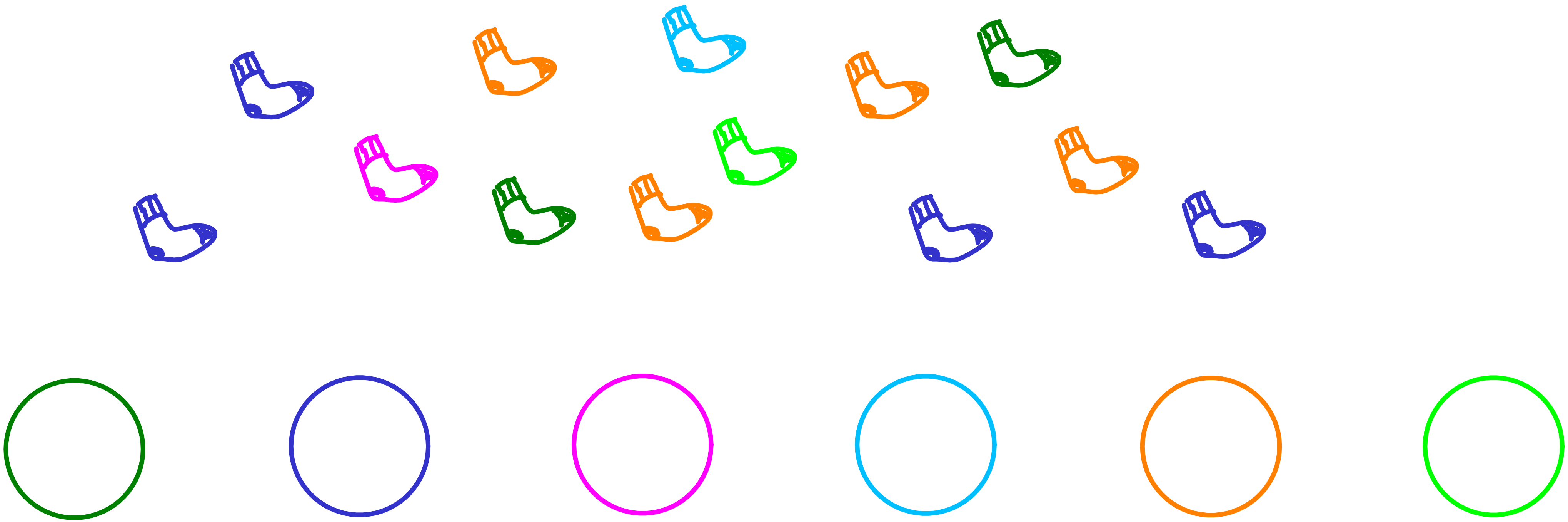

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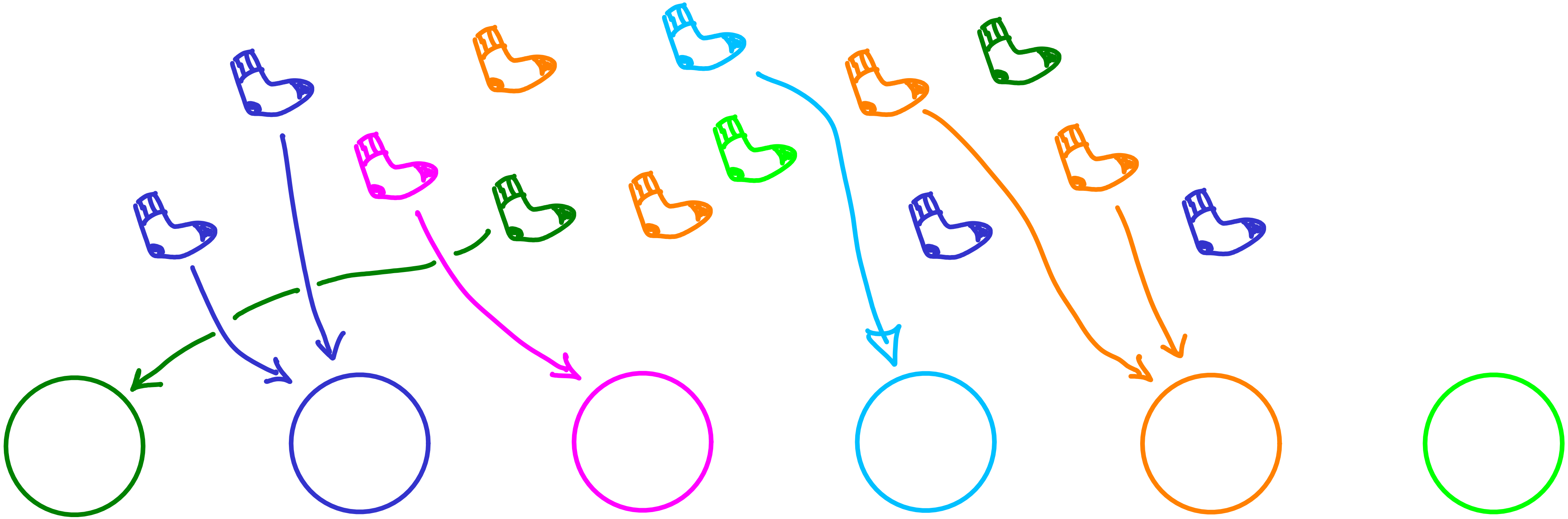

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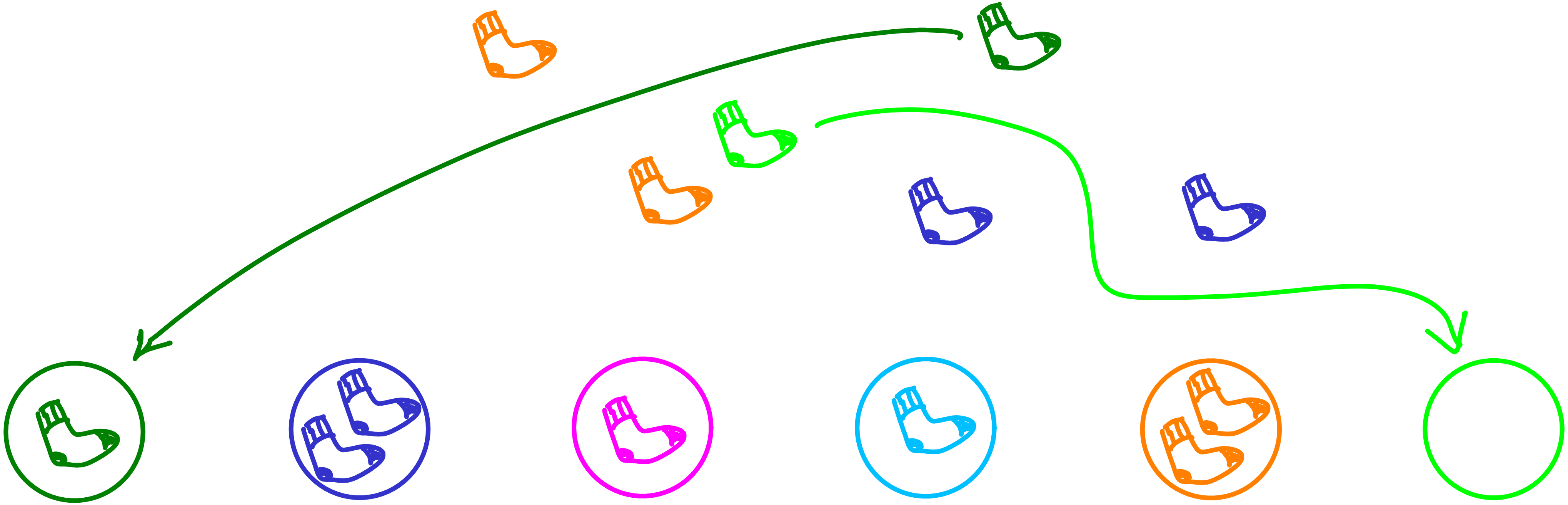

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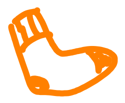
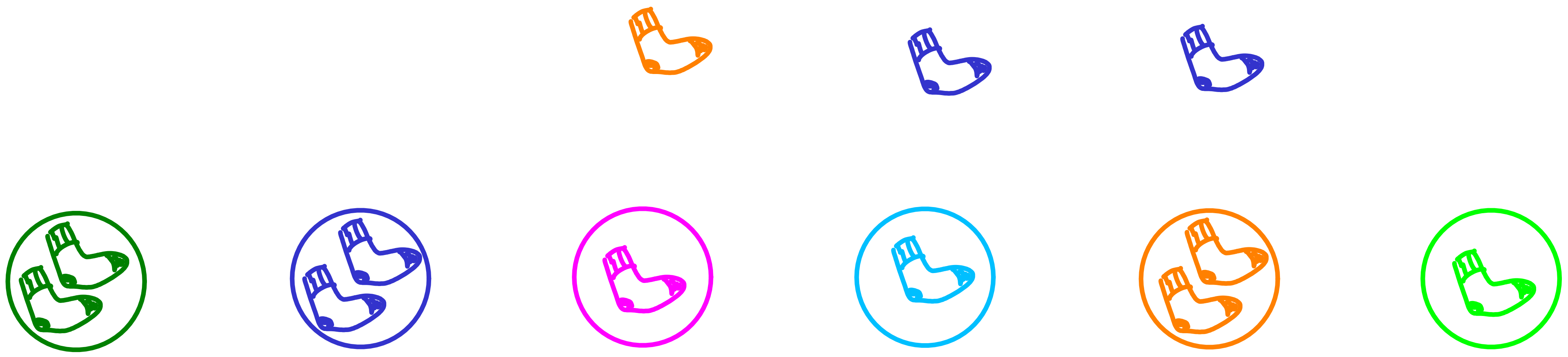


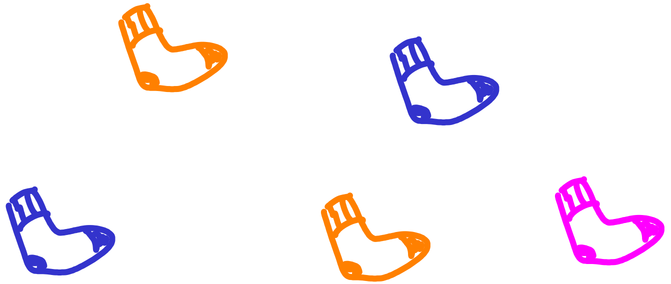
Next pick does it.



THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of n colors, how many do you need to pick (randomly) to get 3 matching?

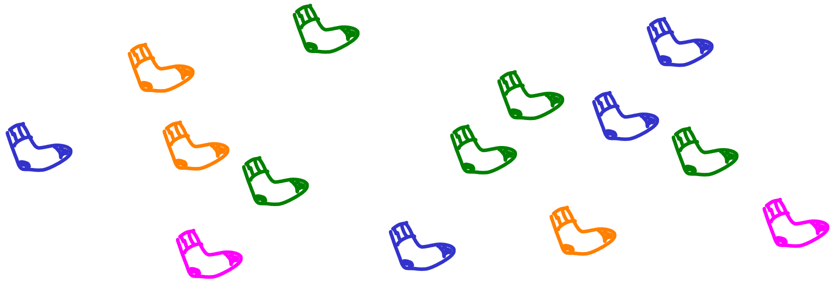
$$\boxed{2n+1}$$



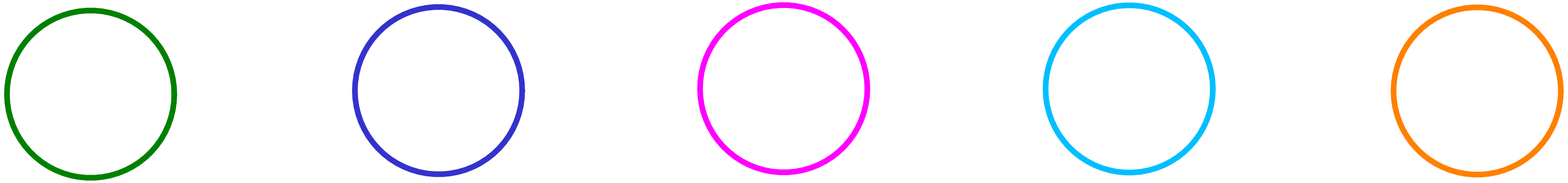
Worst scenario: pick 2 of each color $= \underline{2n}$ before getting a triple.

THE PIGEONHOLE PRINCIPLE

If you have a pile of socks, of n colors, how many do you need to pick (randomly) to get k of one type?



e.g., $n=5$
$k=6$



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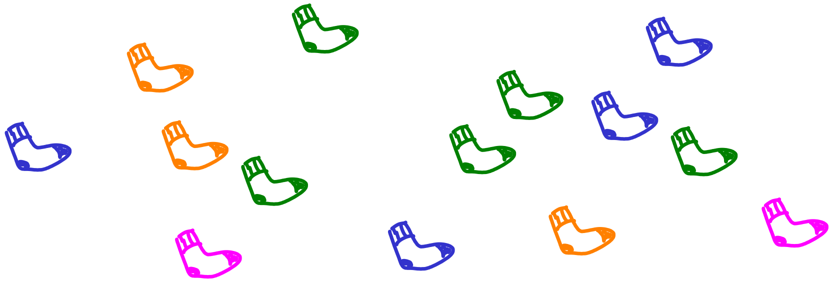
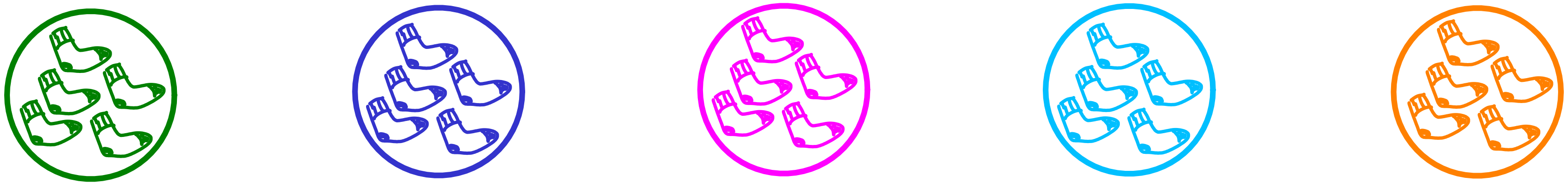


Worst scenario:
pick $k-1$ of each color...



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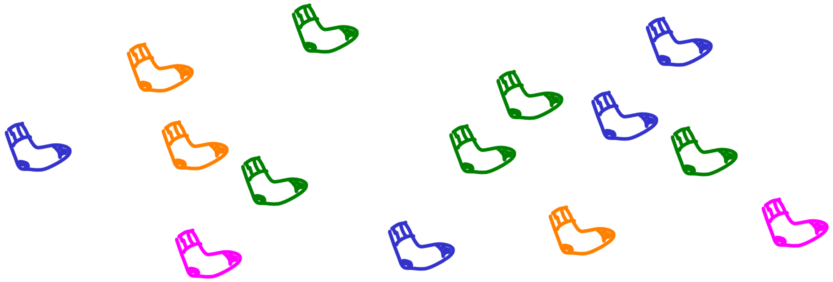
$$\boxed{(k-1)\cdot n+1}$$



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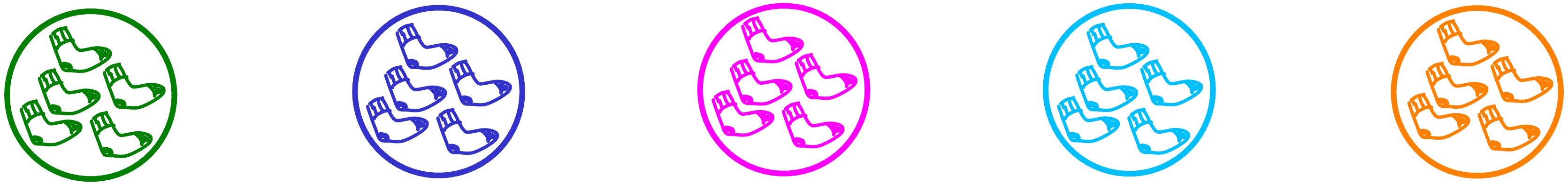
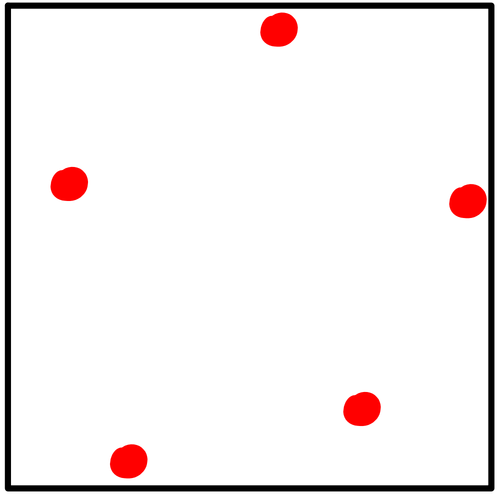


Prove: for any set of 5 points in a unit square,

there are 2 points within distance $\leq \frac{\sqrt{2}}{2}$



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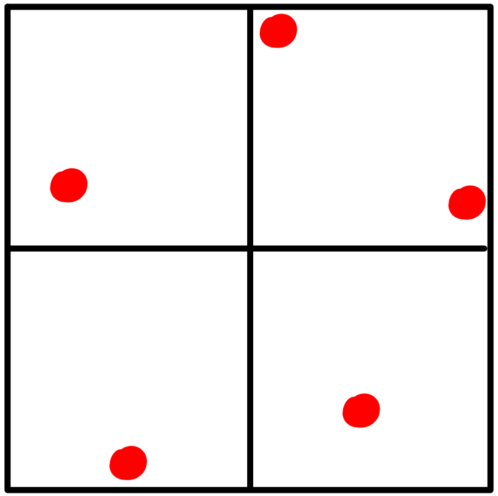

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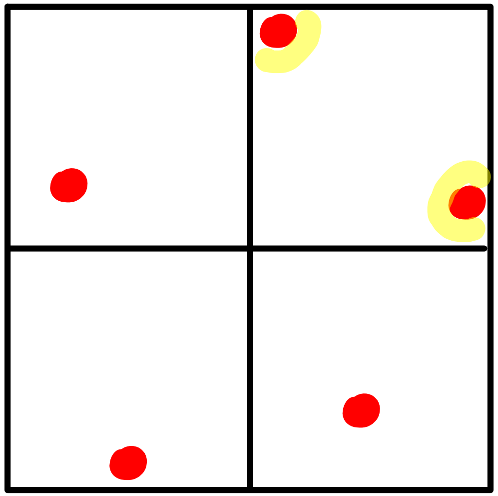

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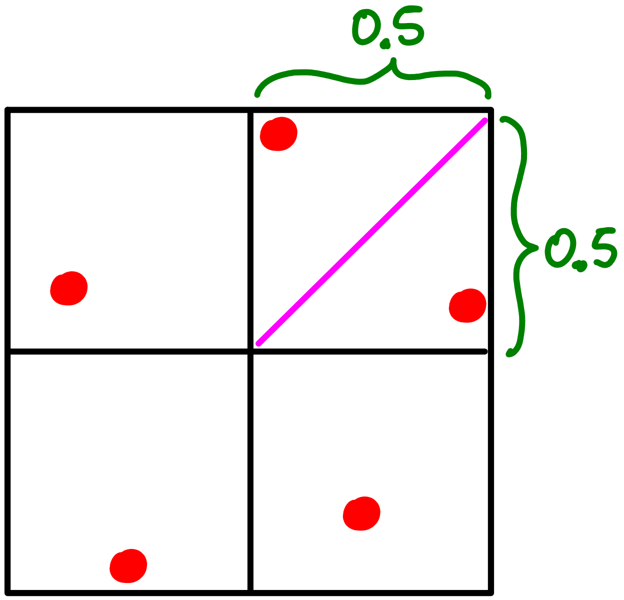

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Max distance in quadrant $= \sqrt{(0.5)^2 + (0.5)^2} = \frac{\sqrt{2}}{2}$ $\square$

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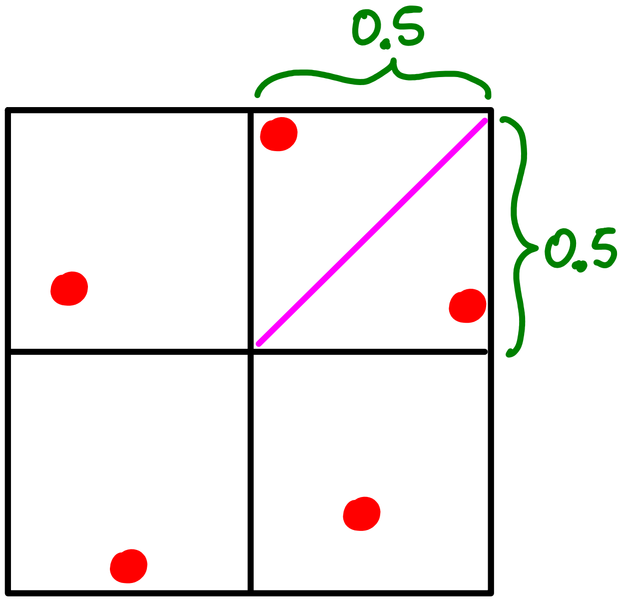

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Works for other shapes & dimensions too.

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n teams = pigeons

$n-1$ possible wins = holes

$\square$

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By pigeonhole, $\left[\begin{array}{l} \text{people} = \text{pigeons} \\ \text{valid \#friends} = \text{holes} \end{array}\right]$ either case works

$\square$

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By pigeonhole, your 51 numbers must include 2 in the same bucket.

$\square$

Let L be a list of 32 8-digit decimal numbers.

31432561

44519287

72911673

$\vdots$

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Range of possible sums? $\rightarrow 0 \ldots \sum_L \rightarrow 0 \ldots \underbrace{32 \cdot 10^8}$

in fact, less than this

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How many subsets can we make? $\rightarrow 2^{32}$

why?

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IN 31432561
IN 44519287
OUT 72911673
etc ⋮

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How many subsets can we make? $\rightarrow$ $\underline{2}^{32} = 4{,}294{,}967{,}296$ "pigeons"

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By pigeonhole, 2 subsets must have the same sum. $\square$

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Note: if 2 subsets have common numbers we can remove them & get a solution with 2 disjoint subsets.

Let L be a list of 32 8-digit decimal numbers.

31432561
44519287
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Proved: there are 2 subsets of L that have the same sum.

For 25-digit numbers, you only need $|L| \geq 90$

(see MCS, ch.15)

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It is difficult to actually find a solution efficiently but it was easy to show that a solution exists.

This type of proof is called "non-constructive"

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This problem is related to applications from shipping/packaging to crypto.

Prove: for every n there are integers $1 \leq a, b \leq 11$ s.t. $a \neq b$, and $\underbrace{10 \mid n^a - n^b}$

10 divides $n^a - n^b$

$$\frac{n^a - n^b}{10} = \text{integer}$$

Prove: for every n there are integers $1 \le a, b \le 11$ s.t. $a \ne b$, and $10 \mid n^a - n^b$

example 1: $n=3$. Pick $a=5$, $b=1$. $3^5 - 3^1 = 240$

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Compute $n^0, n^1, n^2, \ldots, n^{10}$

	0	1	2	3	4	5	6	7	8	9	10
e.g. $n=3$:	1,	3,	9,	27,	81,	243	729,	2187,	6561,	19683,	59049

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Compute $n^0, n^1, n^2, \ldots, n^{10}$ and place each in a bucket by last digit.

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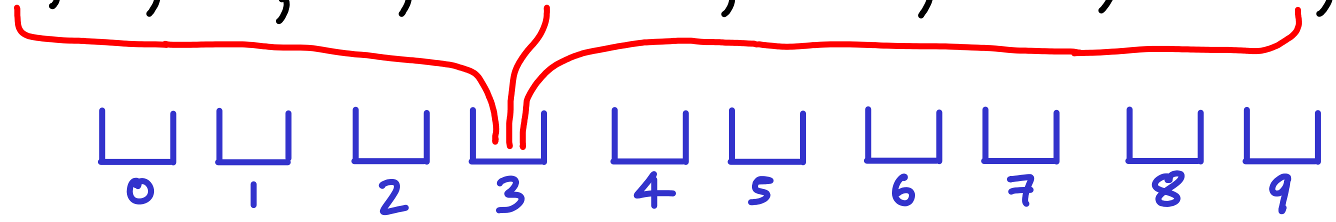

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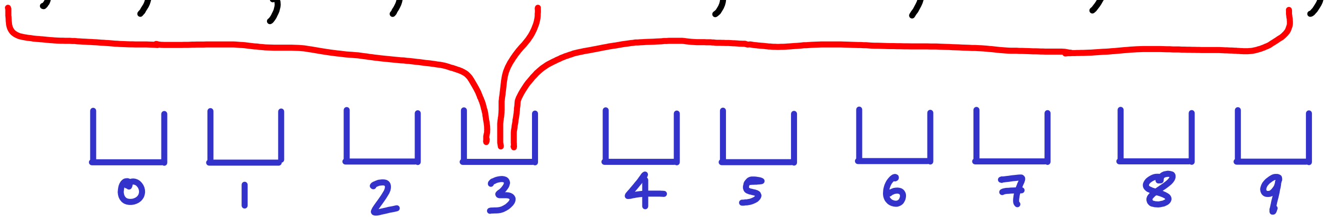
example 3: $n=17$. Pick $a=6$, $b=2$. $17^6 - 17^2 = 24{,}137{,}280$

We want $n^a - n^b$ to end with a 0. $\rightarrow$ n^a & n^b end with same digit.

Compute $n^0, n^1, n^2, \ldots, n^{10}$ and place each in a bucket by last digit.

$$\text{e.g. } n=3: \quad \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1, & 3, & 9, & 27, & 81, & 243 & 729, & 2187, & 6561, & 19683, & 59049 \end{array}$$



Proved by pigeonhole. 11 powers (pigeons), 10 buckets (holes) $\square$

FILE ZIPPING Objective: reduce storage (#bits) for any given file

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files A → zip via any single zipping algorithm → A.zip

B → B.zip

BIG ideally small

FILE ZIPPING

Objective: reduce storage (#bits) for any given file

files A, B $\rightarrow$ zip via any single zipping algorithm $\rightarrow$ A.zip, B.zip $\rightarrow$ unzip when needed $\rightarrow$ A, B

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If every n-bit file zips to a $<n$-bit file, ...?

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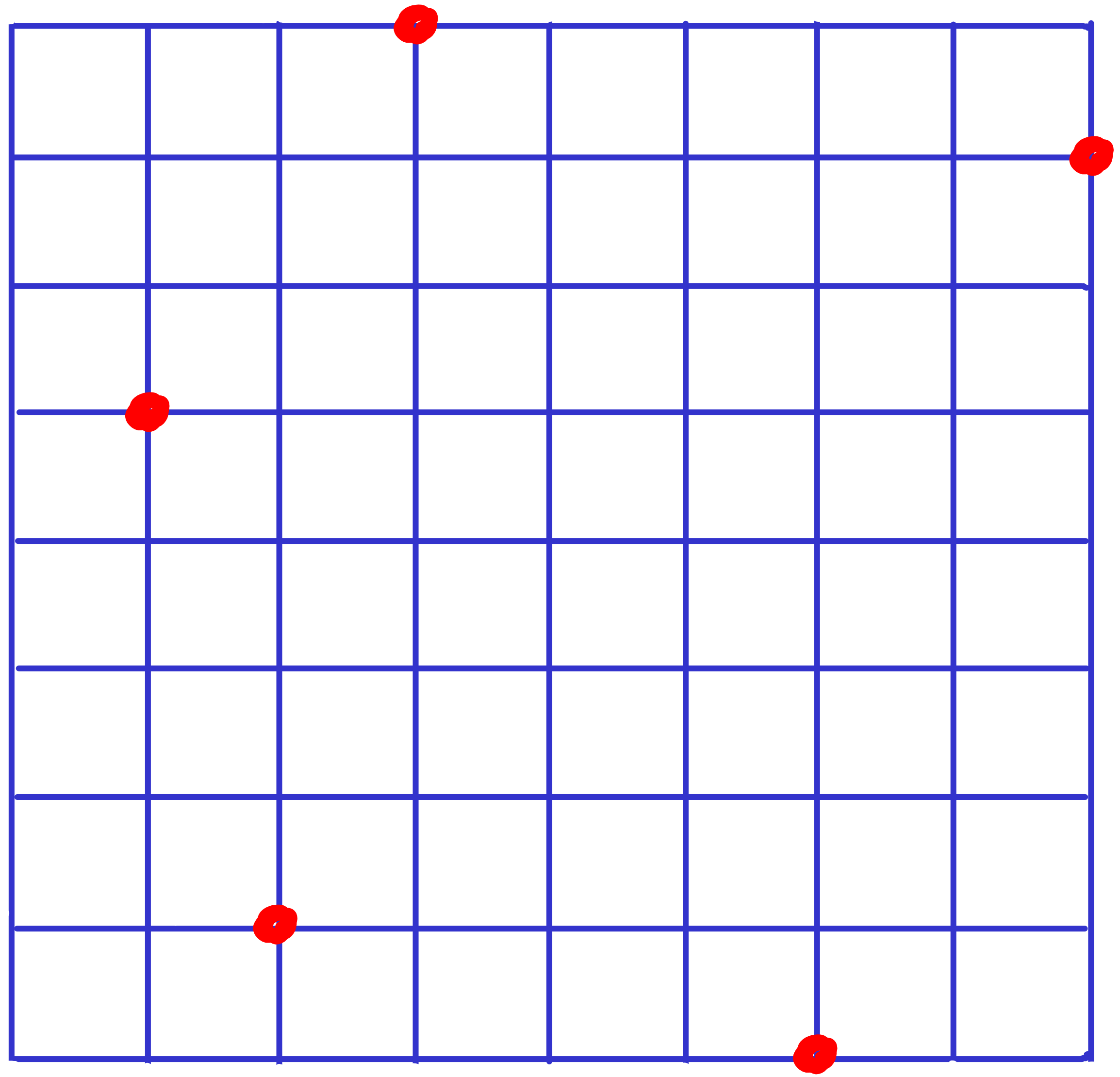
!

If every n-bit file zips to a $<n$-bit file, $\exists$ A,B s.t. A.zip = B.zip $\square$

Consider 5 points on a grid.
(size & dimensions don't matter)

example:



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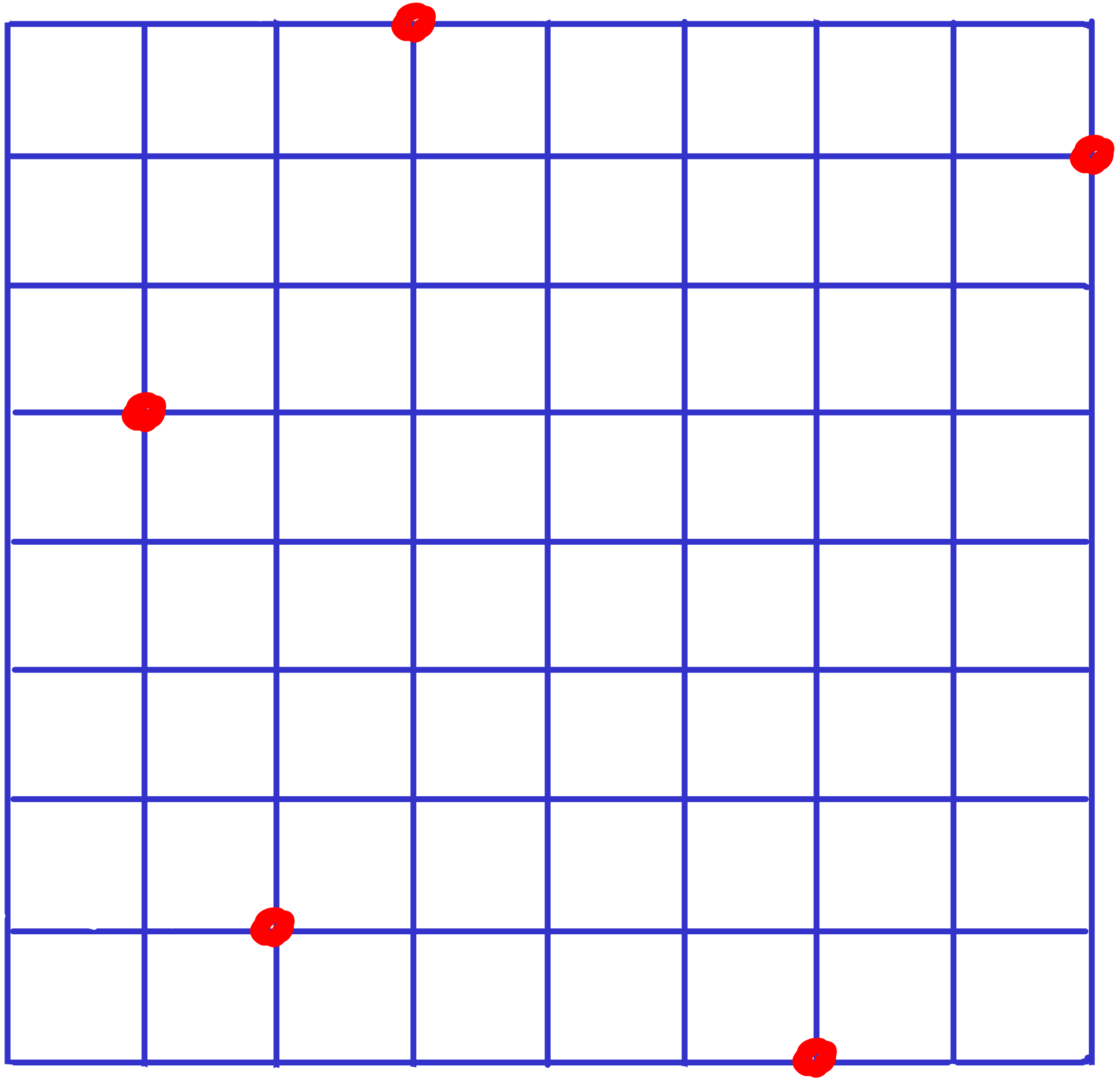
Prove: $\exists$ 2 points such that their midpoint is on the grid.



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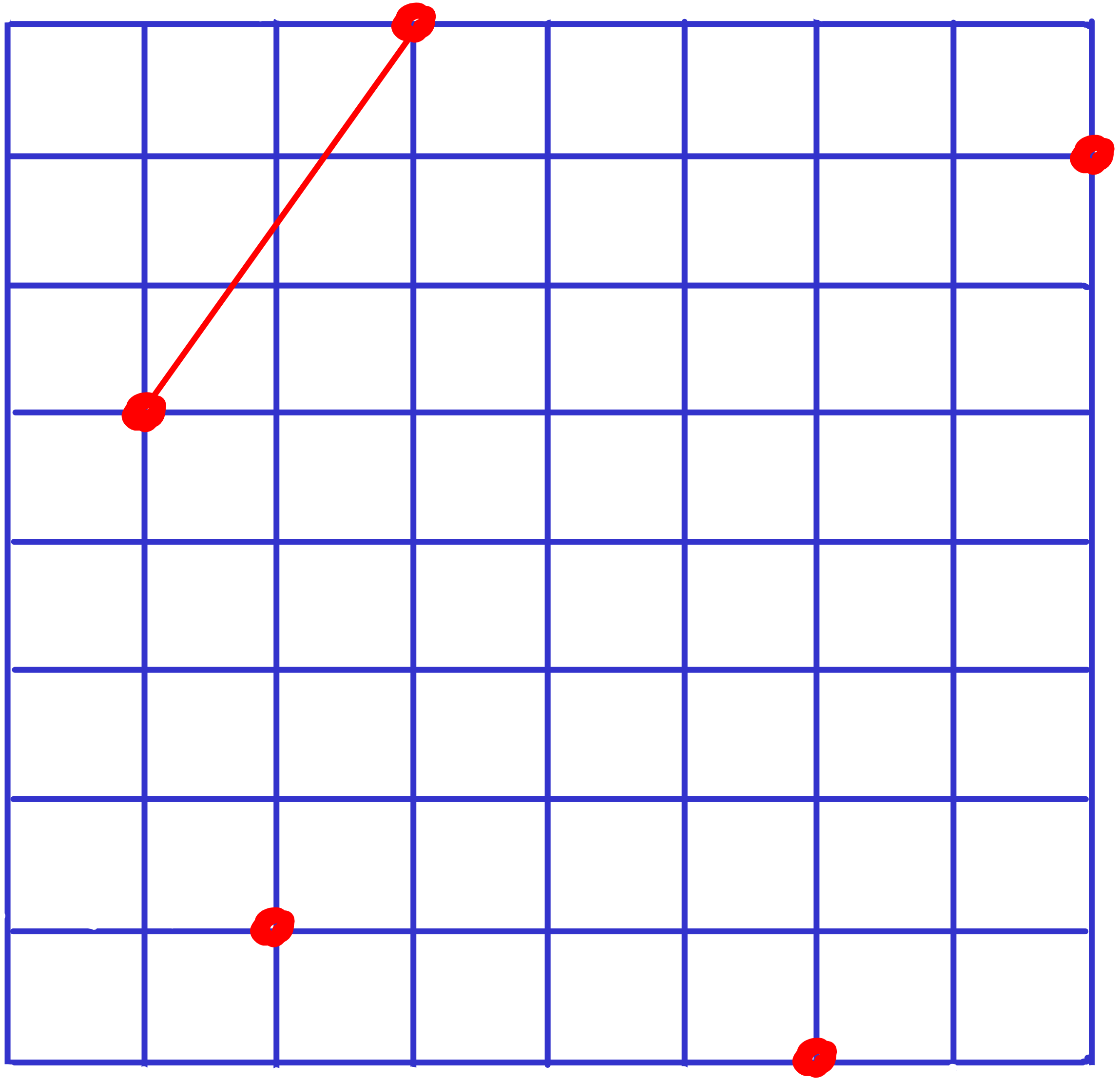
X segment doesn't pass through grid point



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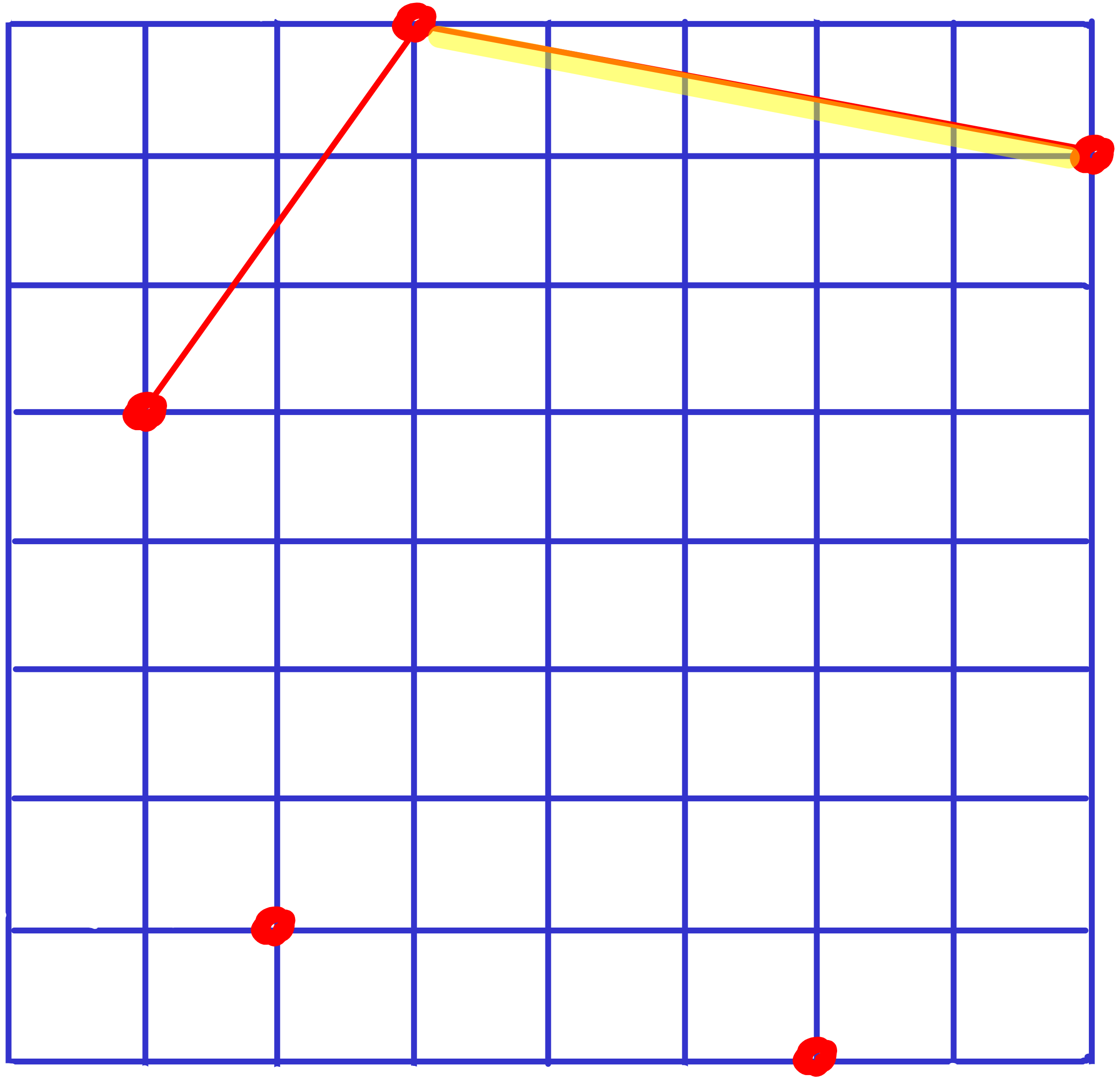

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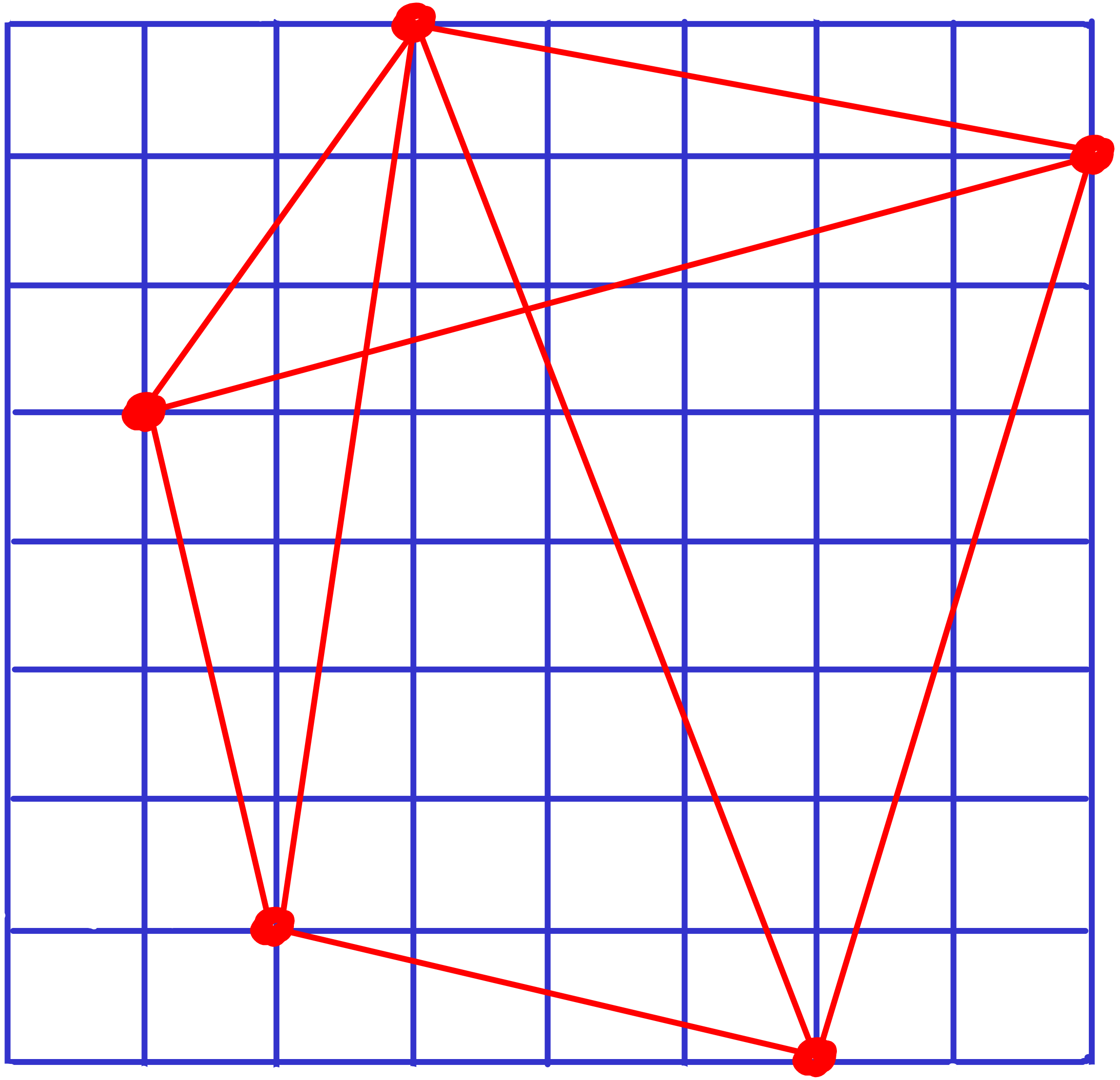

✗ segments don't pass through grid point

Consider 5 points on a grid.
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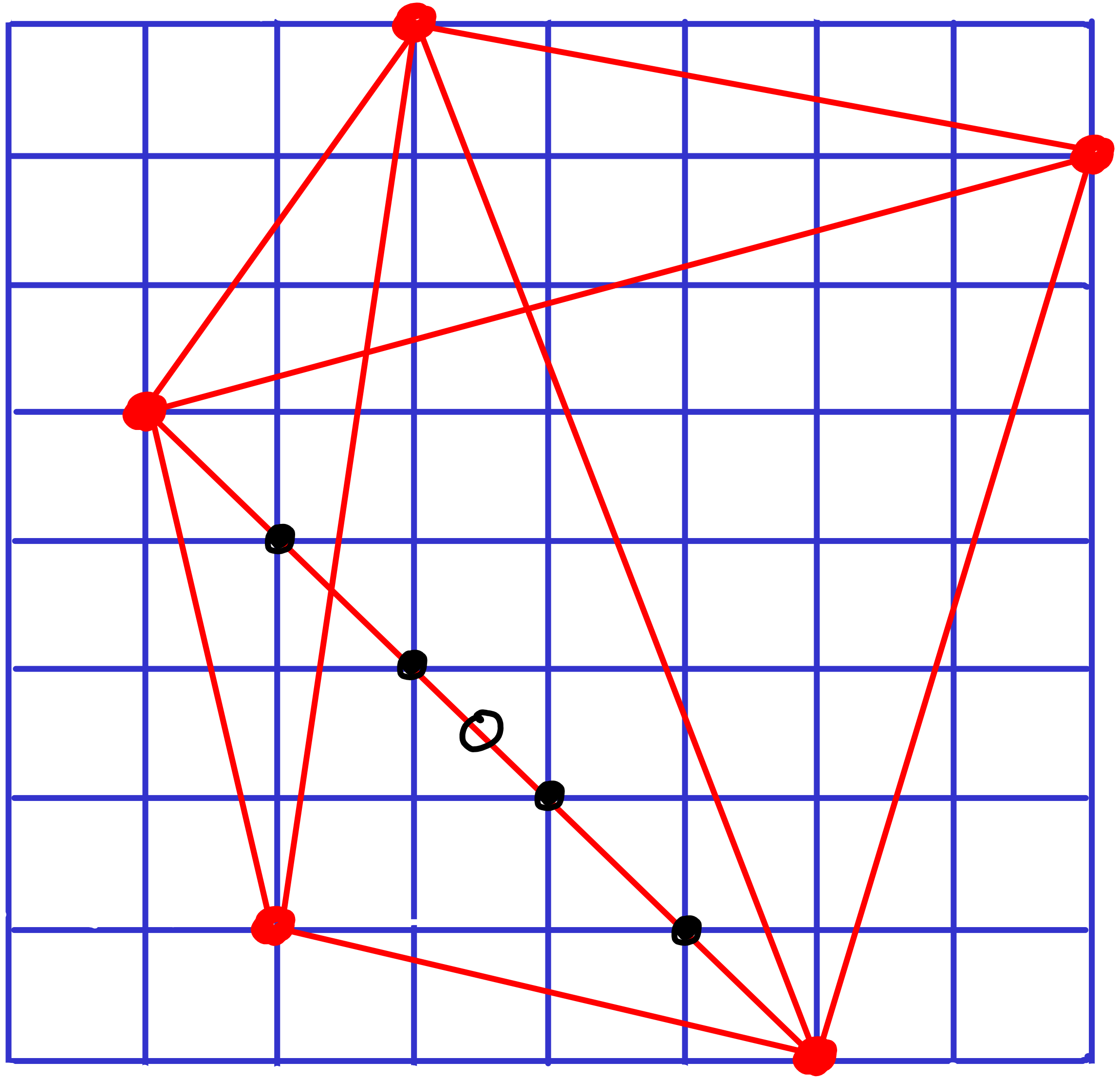
Example:

× segment does pass through grid points but midpoint isn't on grid.



Consider 5 points on a grid.
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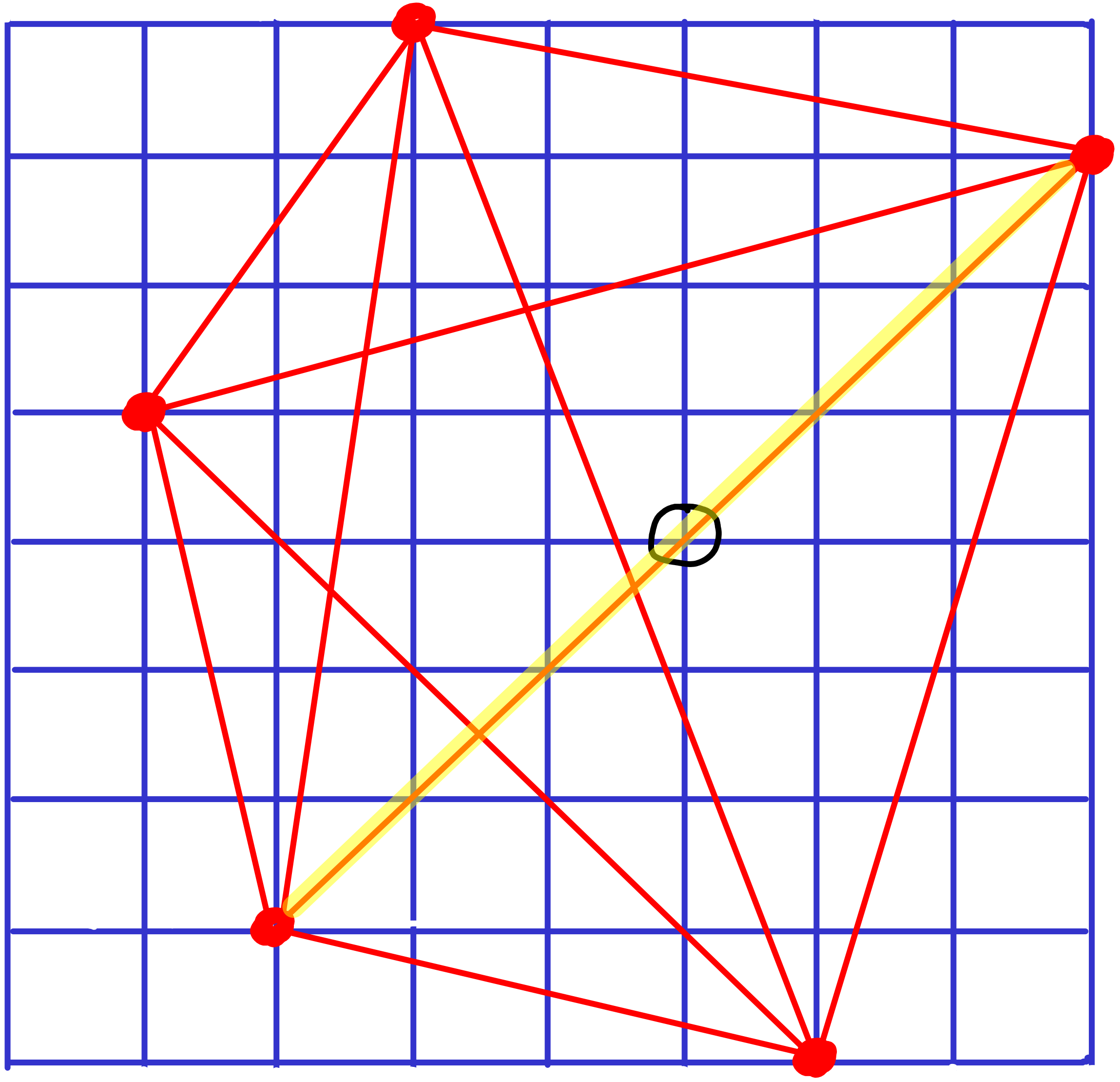
Prove: ∃ 2 points such that their midpoint is on the grid.

Example:

✓ success

(try to shift points & avoid the midpoint claim)



Prove: among any 5 grid points,
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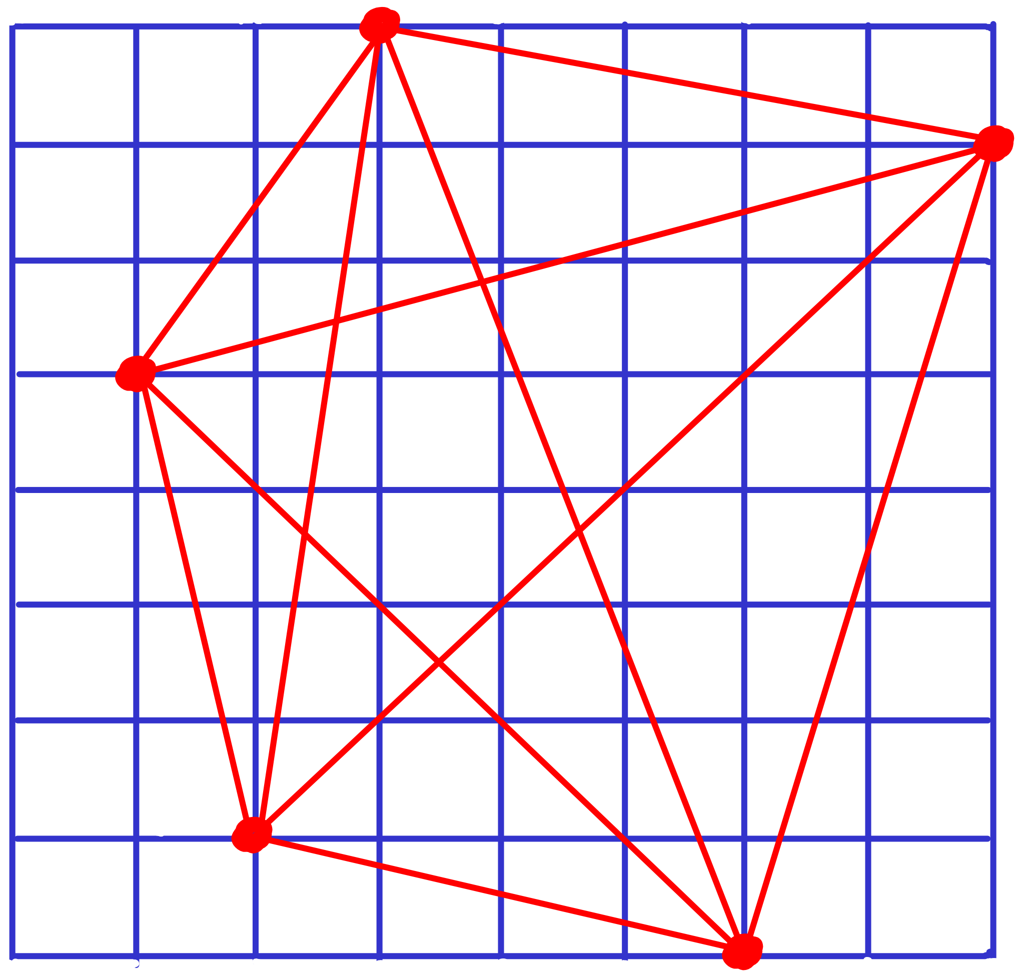

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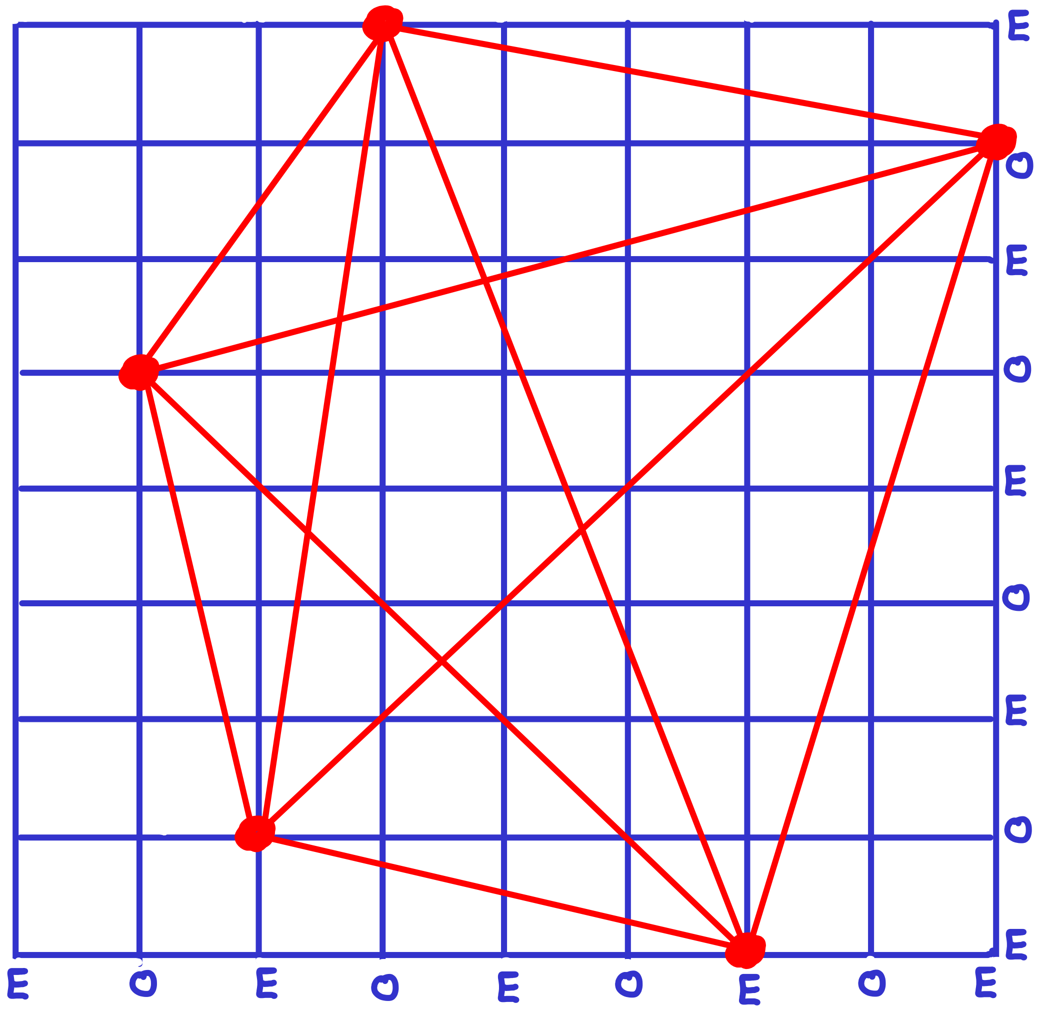
odd v Even



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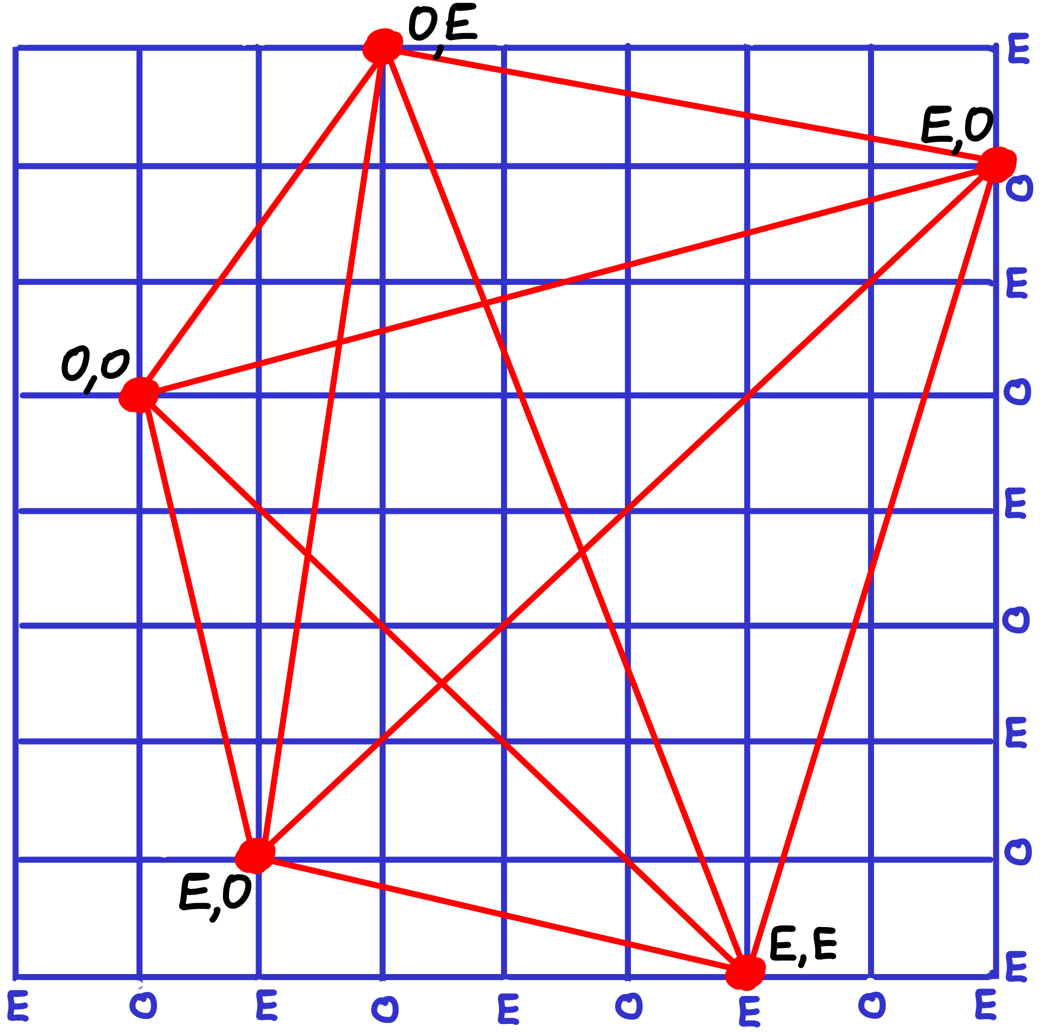
4 grid position types for (x,y):

odd v Even: (O,E) (E,O) (O,O) (E,E)



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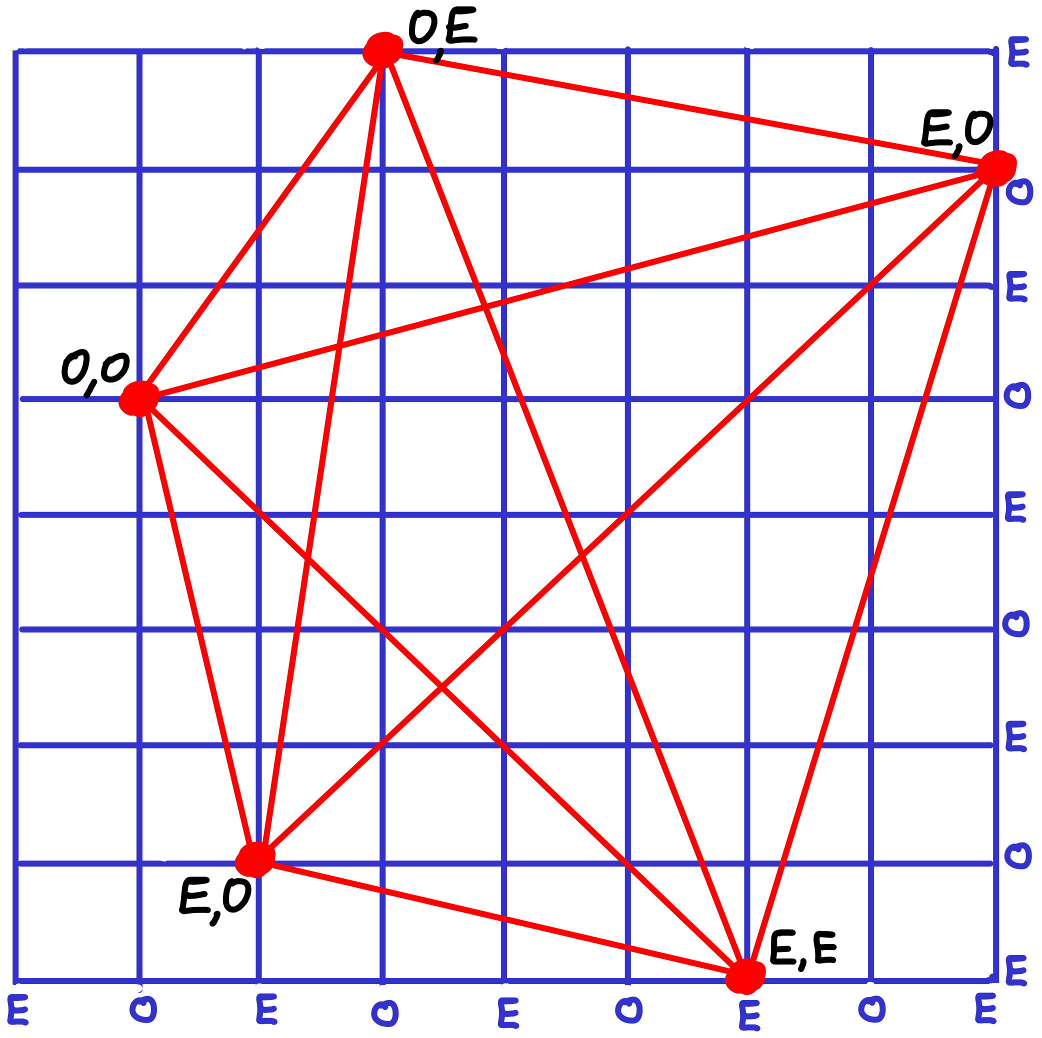

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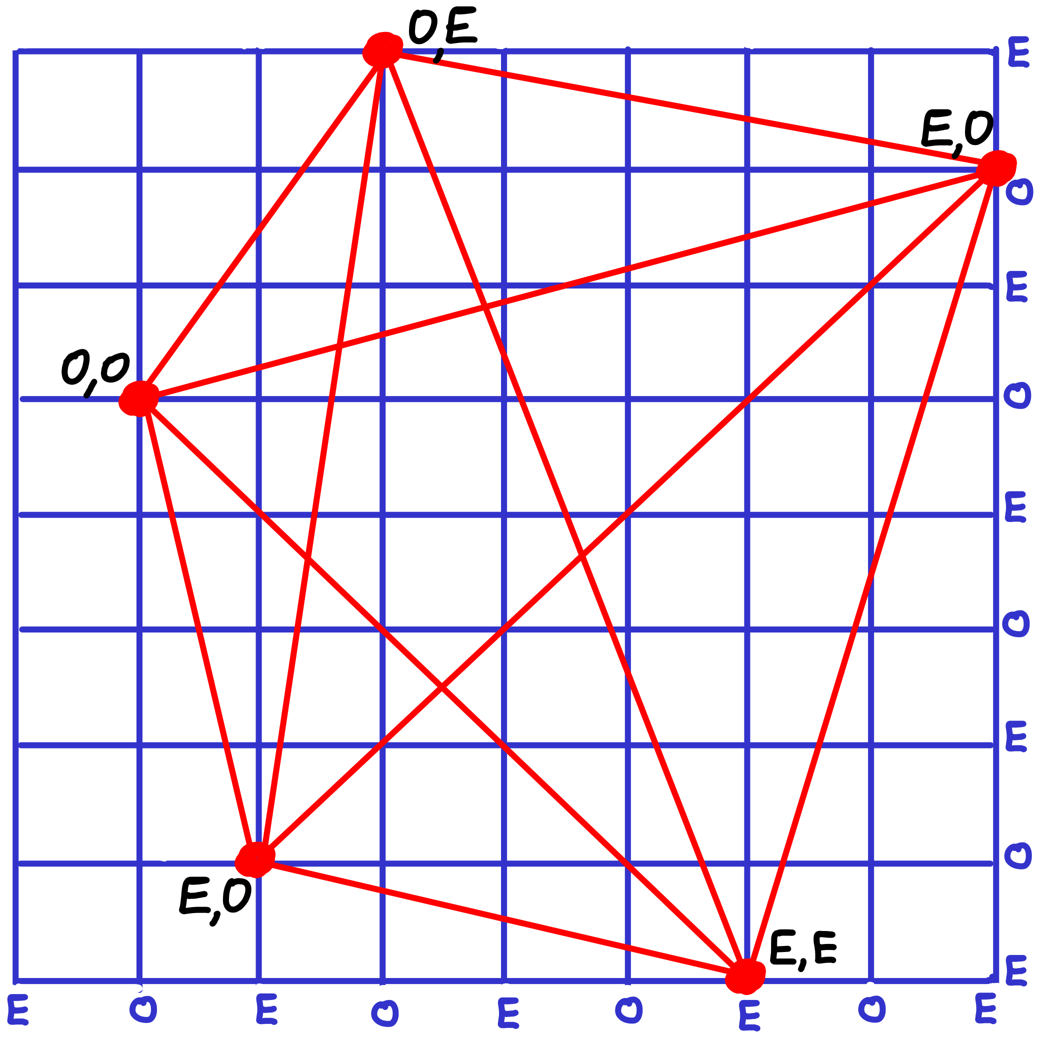
For each coordinate $c = \{x,y\}$
$$\text{midpoint}(A,B)_c = \frac{1}{2}(A_c + B_c)$$



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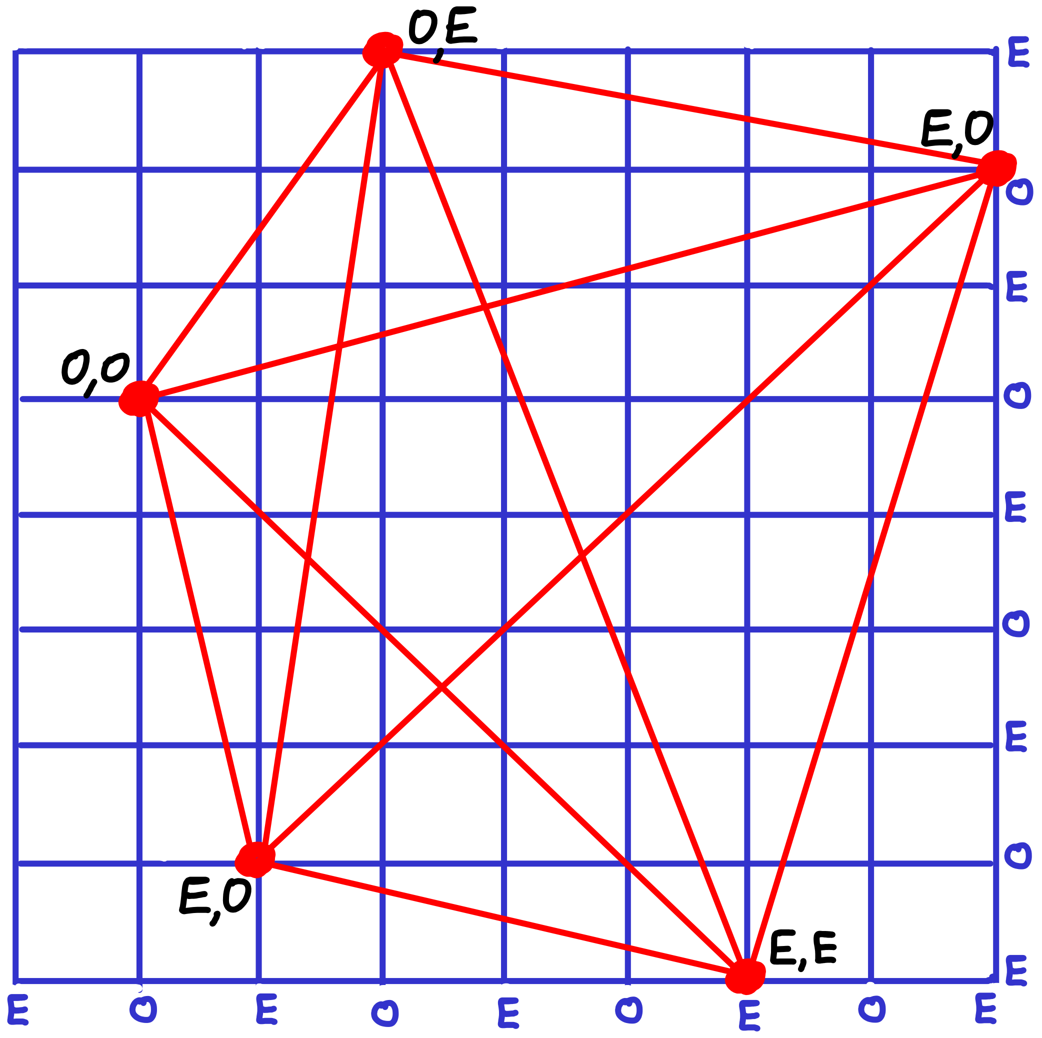

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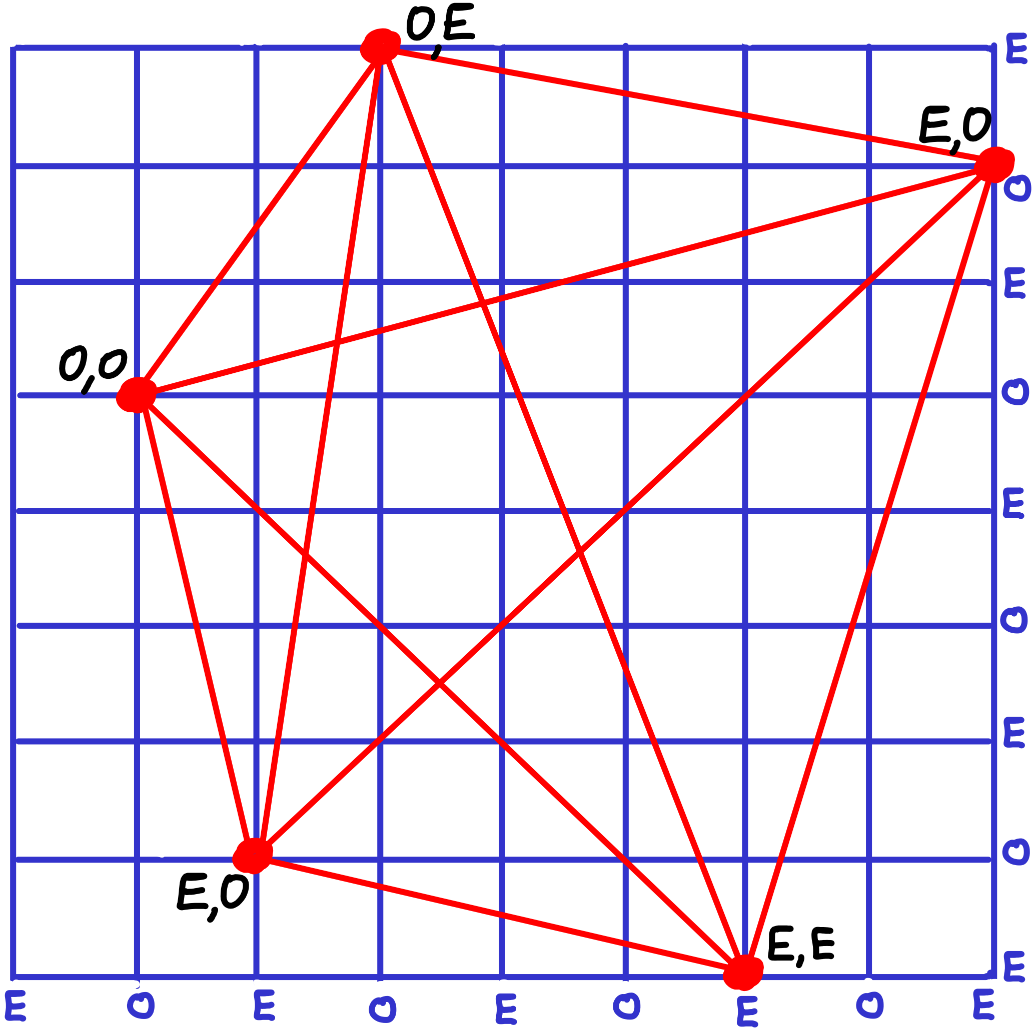

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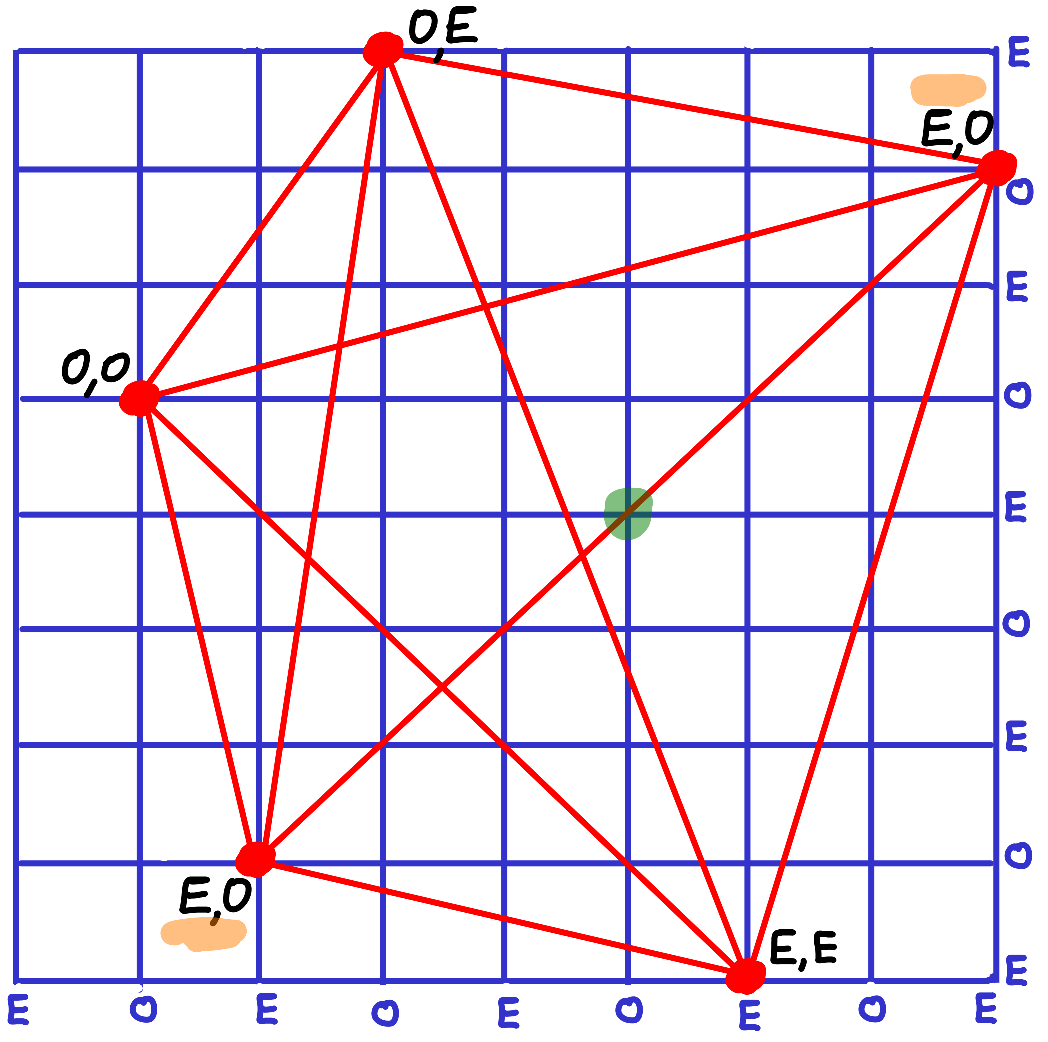
So midpoint $= \frac{1}{2} 2k =$ integer $\square$



sequence of distinct numbers

17 3 6 92 8 22 31 27 18 13 45 68 33 72 49

Given a sequence of distinct numbers, a subsequence of size k is k-monotone if it is either increasing or decreasing.

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$$\underbrace{17 \quad 3}_{\text{2-monotone}} \quad 6 \quad 92 \quad 8 \quad 22 \quad 31 \quad 27 \quad 18 \quad 13 \quad 45 \quad 68 \quad 33 \quad 72 \quad 49$$

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5-monotone

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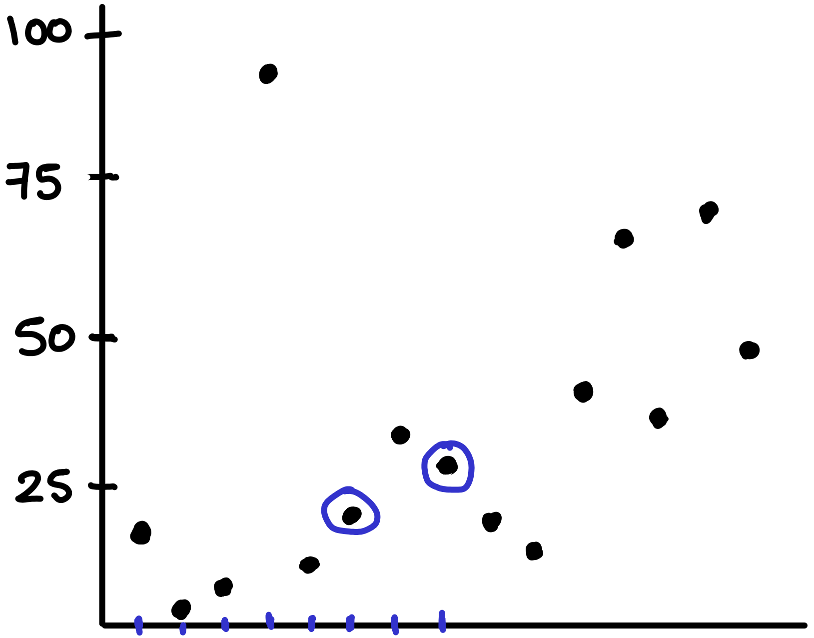
$(6,22)$ $(8,27)$

Geometric view:

Create 2D point set



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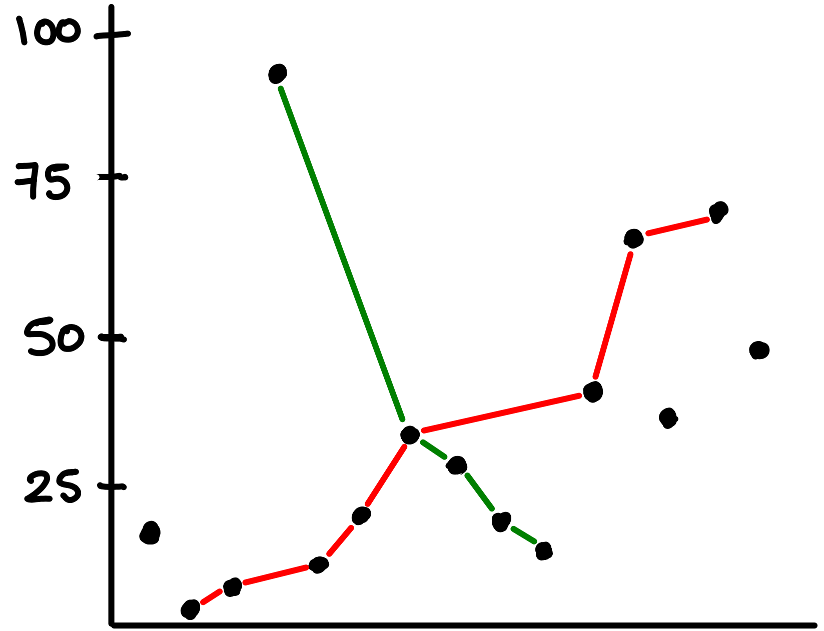
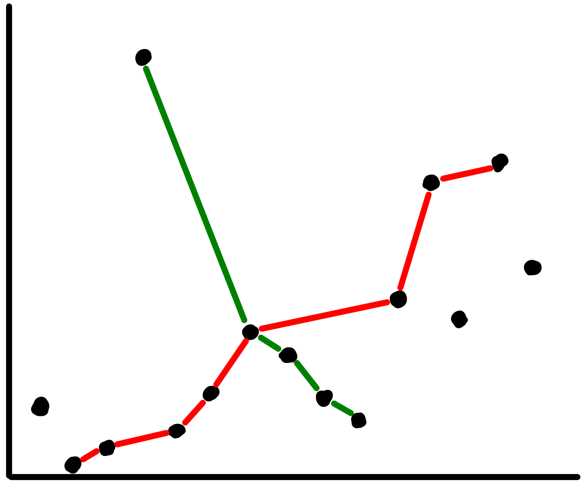


How large must a sequence be,
to guarantee a "large" monotone subsequence?



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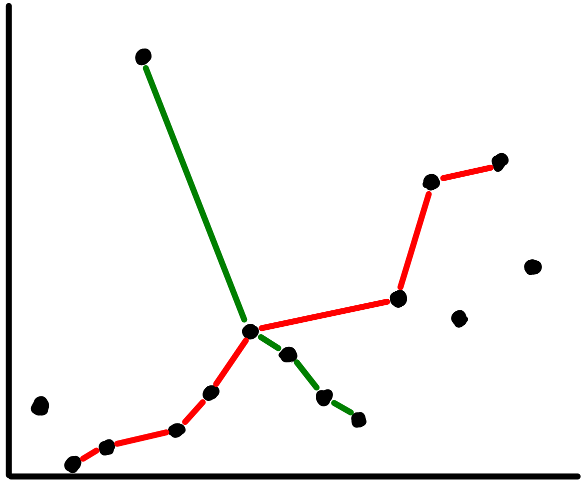
Claim: A $(n+1)$-monotone subsequence exists in every sequence of size n^2+1.



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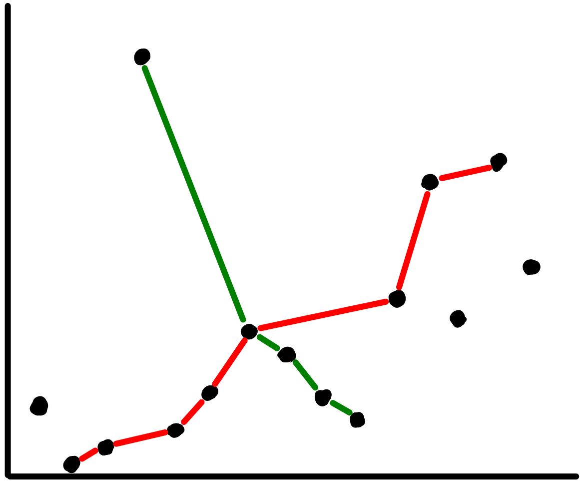

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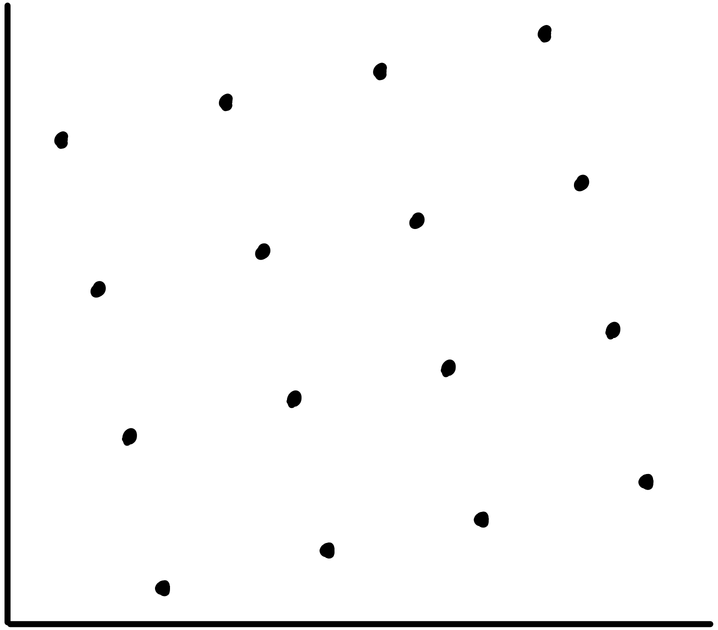
$n=4 \to n^2 = 16$ pts:



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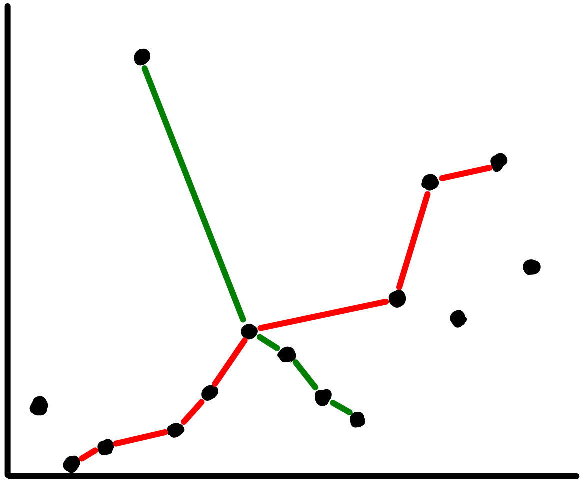

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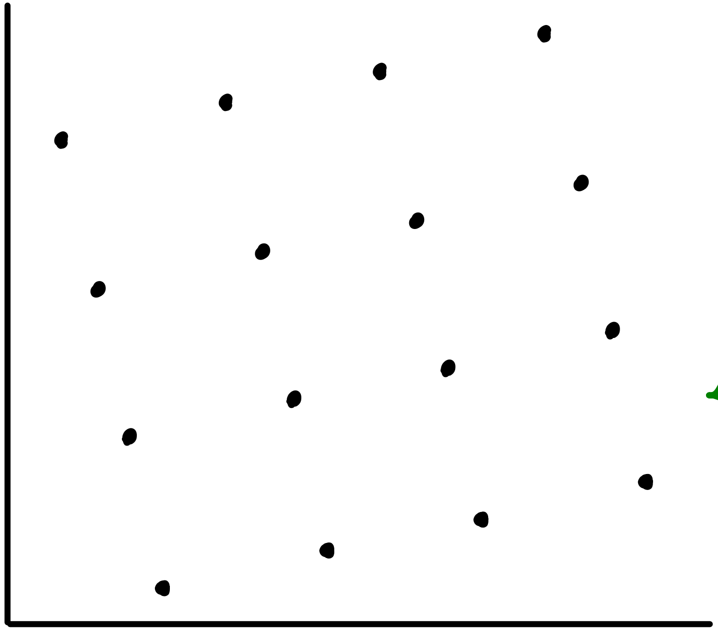

With 1 more point, we can. *

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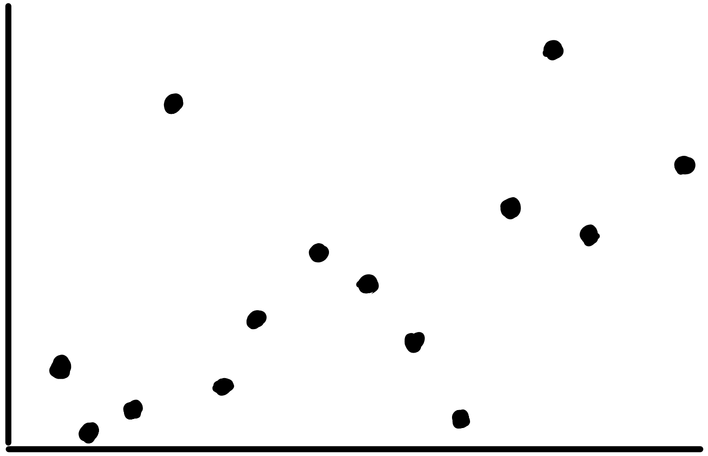

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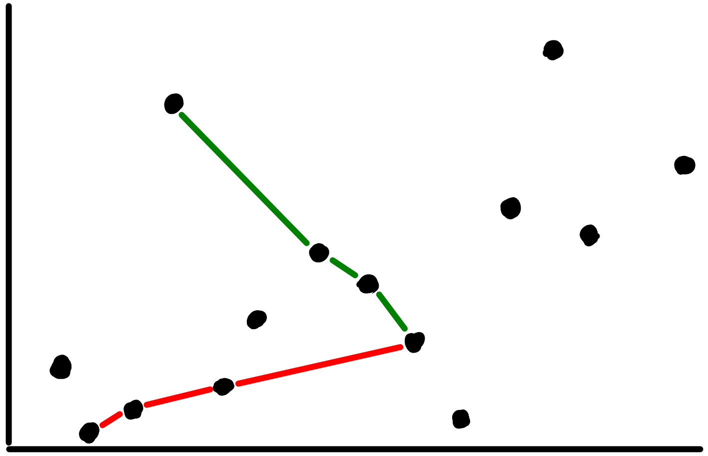
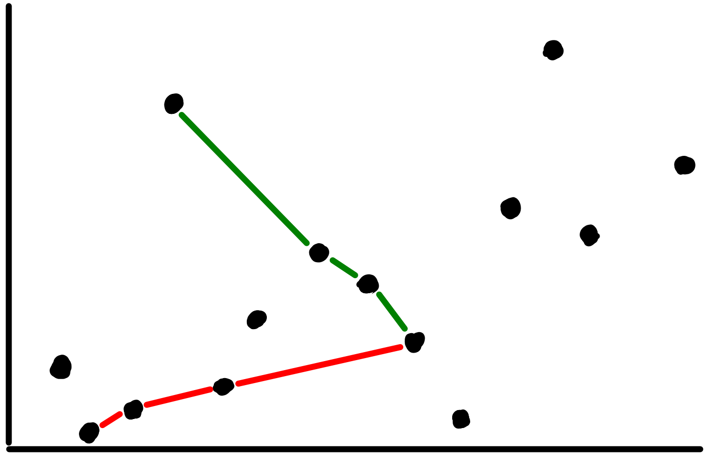


Define u_i, d_i : sizes of longest upward & downward chains ending at position i.

Prove: A $(n+1)$-monotone subsequence exists in every sequence of size n^2+1.



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Assuming no $(n+1)$-monotone chain, for all i : $u_i, d_i \leq n$

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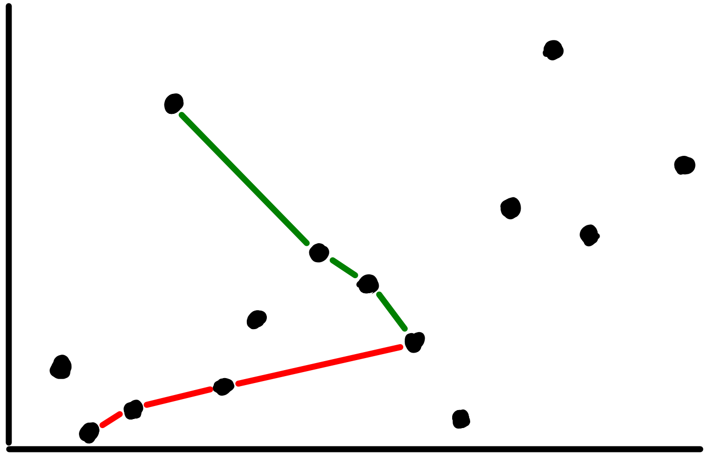

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By pigeonhole, 2 of the $\underbrace{n^2+1 \text{ points}}_{\text{pigeons}}$ have equal (u,d).

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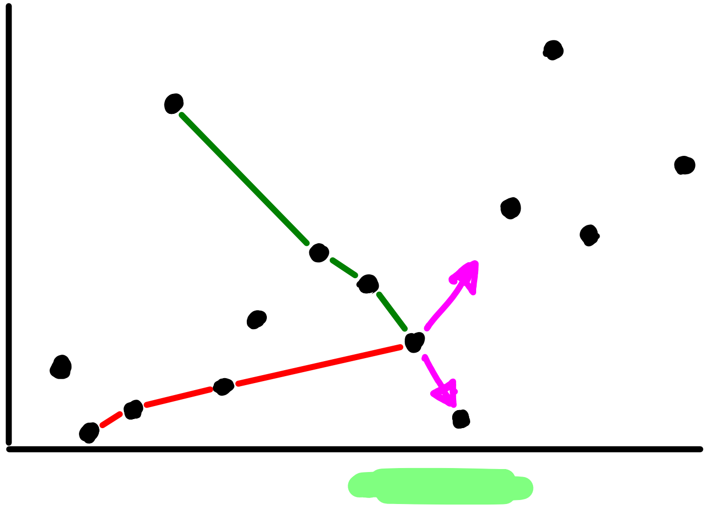

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But...

position $j = i+k$ will extend u_i or d_i $\rightarrow$ either $u_j > u_i$ or $d_j > d_i$

$(k>0)$

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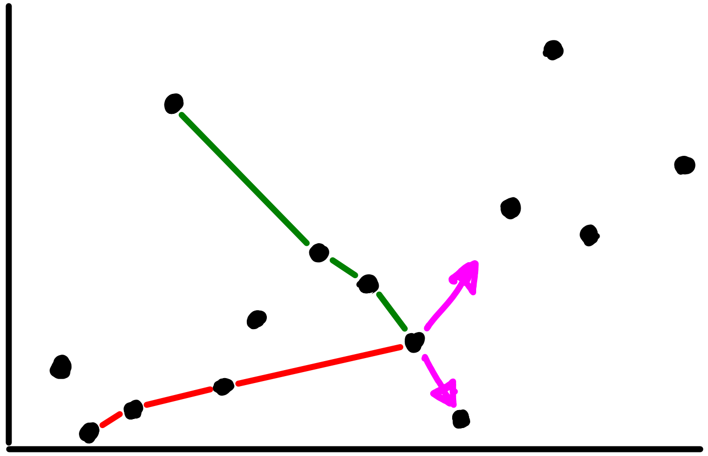

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↳ So for all i,j: $(u_i, d_i) \neq (u_j, d_j)$

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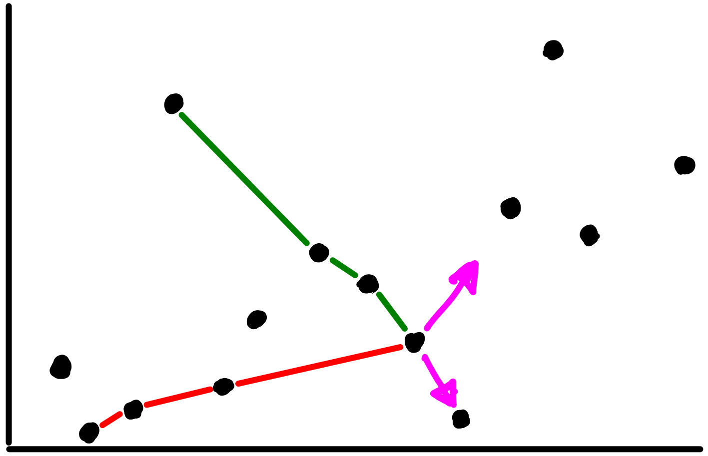

Define u_i, d_i : sizes of longest upward & downward chains ending at position i.

Assuming no $(n+1)$-monotone chain, for all i: $u_i, d_i \leq n \Rightarrow \leq n^2$ holes

By pigeonhole, 2 of the $\underbrace{n^2+1 \text{ points}}_{\text{pigeons}}$ have equal (u,d).

But...

position $j = i+k$ will extend u_i or d_i $\rightarrow$ either $u_j > u_i$ or $d_j > d_i$

↳ So for all i,j: $(u_i, d_i) \neq (u_j, d_j)$ Contradiction. □