RANDOM VARIABLES

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A quote from the Scheinerman textbook:

"A random variable is neither random nor variable"

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We can also eliminate absurd events, e.g., $P(X=13)=0$

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$$E(X) = \sum_{*} y \cdot P(X=y)$$ see example 34.3 in book (mainly p.237) (Scheinerman)

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$$E(x) = \frac{0 + 10 + 16 + 18 + 16 + 10}{36}$$

(~ 1.944)

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$$E(X + Y) = E(X) + E(Y)$$

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$c_1, c_2 \in \mathbb{R}$

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for all a,b ...

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2 dice, A, B. $\quad X$ = result of A. $\quad Y$ = result of B. $\quad Z = X+Y$

$E(Z) = E(X+Y) = E(X) + E(Y) = 2 \cdot 3.5 = 7$

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If X & Y are independent, then $E(X \cdot Y) = E(X) \cdot E(Y)$

However, $E(X \cdot Y) = E(X) \cdot E(Y)$ does NOT imply X & Y are independent.

(see example 34.15)
(Scheinerman)

- We are skipping the proofs of most statements in this section.

- You are not required to study these, but it would probably be beneficial.

- We are also skipping **variance**, which is an important concept to learn independently.