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A: yes, 5: $4 < 5 < 6$, and $5 = 2 \cdot 2 + 1$ (is odd)

Here the answer contains an example that can be verified

Sometimes we can prove that something exists without constructing an example or verification method.

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Claim: at some time t', $0<t'<1$, your speed will be zero.

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We don't know where this happens, but it must happen.

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- At any given instant, there exist two positions on the equator that are diametrically opposite

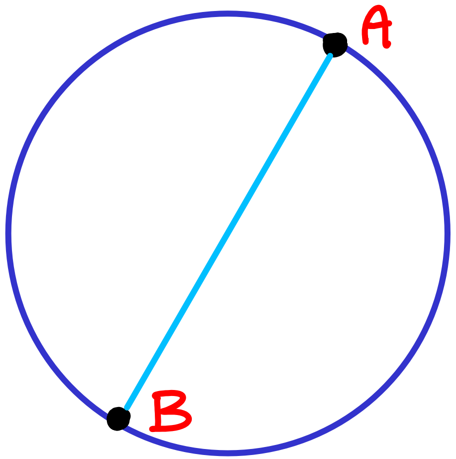
  (the center of the planet is exactly in between)

  and that have exactly the same temperature.

  (Assume perfectly spherical planet.)

...

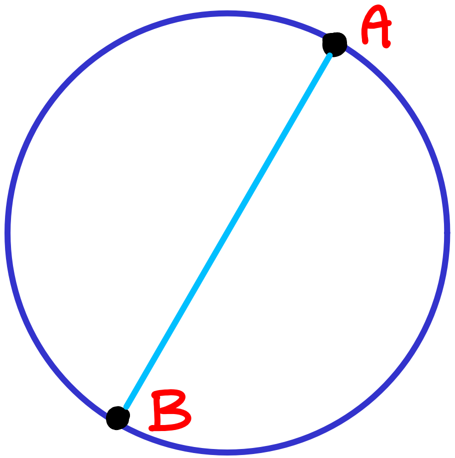
Consider points A, B, diametrically opposite.



Consider points A, B, diametrically opposite.

If temperatures are equal, done.

Else, without loss of generality, $t(A) > t(B)$.



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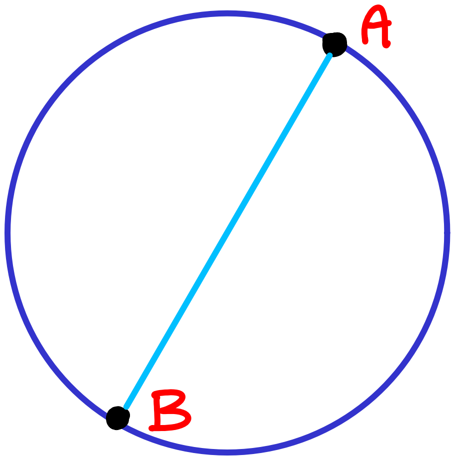
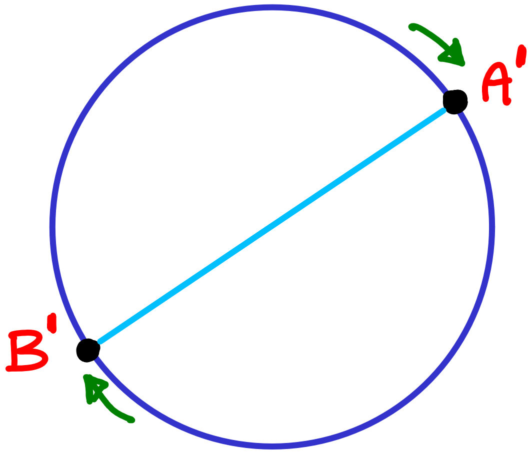
Then move clockwise, staying opposite. Track points A', B'.

All of this applies to one instant in time.





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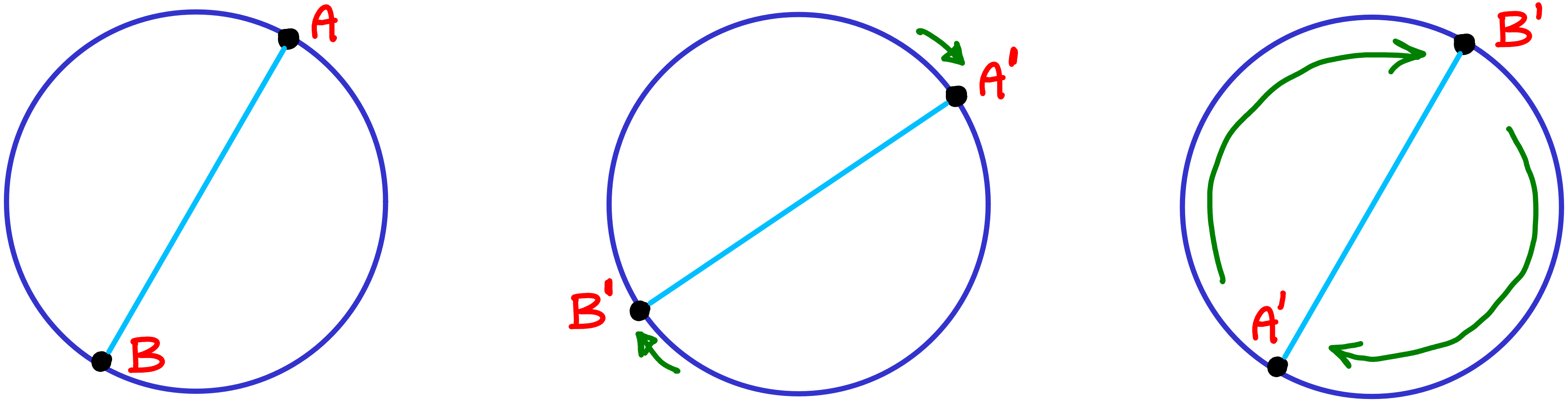

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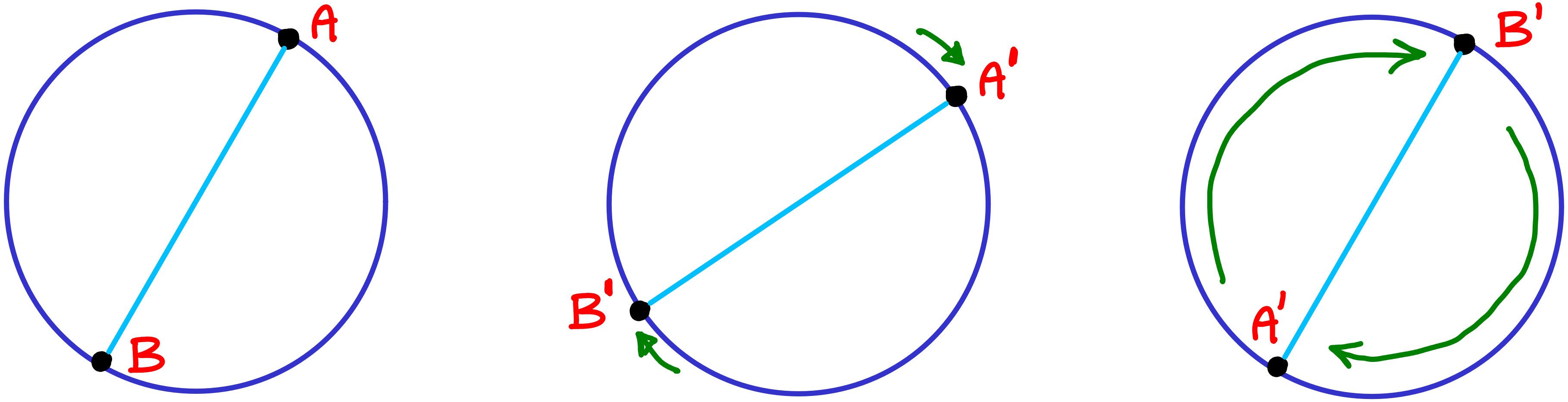
Then move clockwise, staying opposite. Track points A', B'.

Continue while $t(A') > t(B')$. But when $B' = A$, $t(B') > t(A')$

So somewhere before rotating 180°, we had $t(A') = t(B')$. □



Note that there is no way to tell where the two points are.

More amazing fact: (Borsuk-Ulam theorem)

At any time there are 2 diametric points somewhere on the planet (not necessarily on the equator) that have equal temperature AND equal pressure.

(works for any 2 continuous functions)

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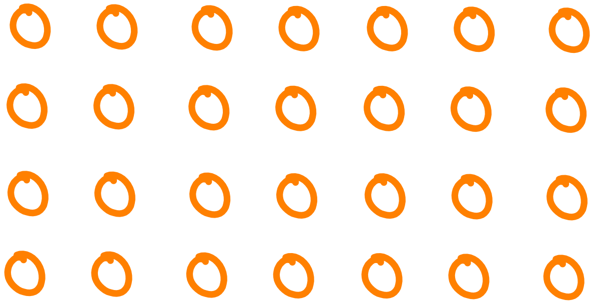
$a = \sqrt{2}^{\sqrt{2}}$, $b=\sqrt{2}$. $\square$

This proof doesn't tell us exactly what a is, but it exists.

The game of Chomp

2 players take turns eating cookies from a grid.



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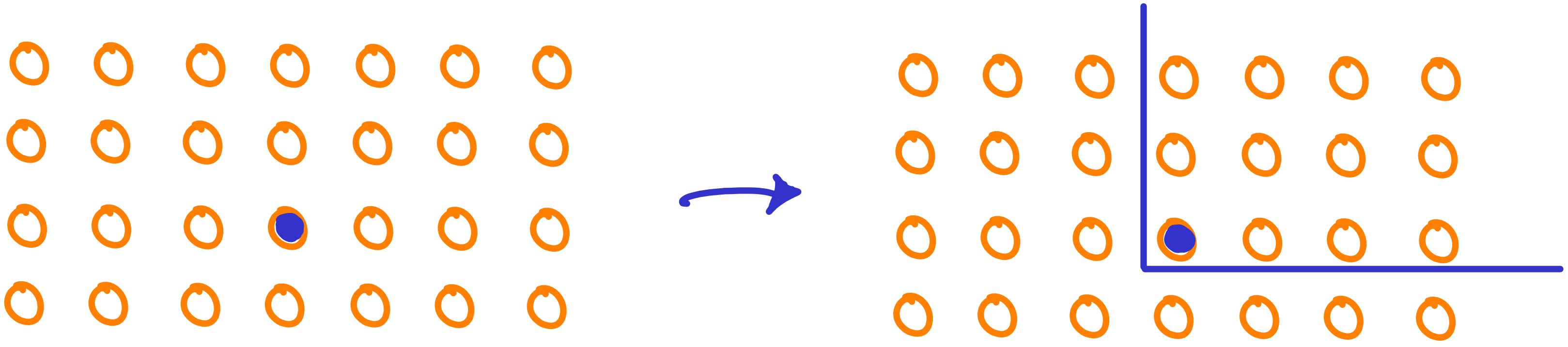

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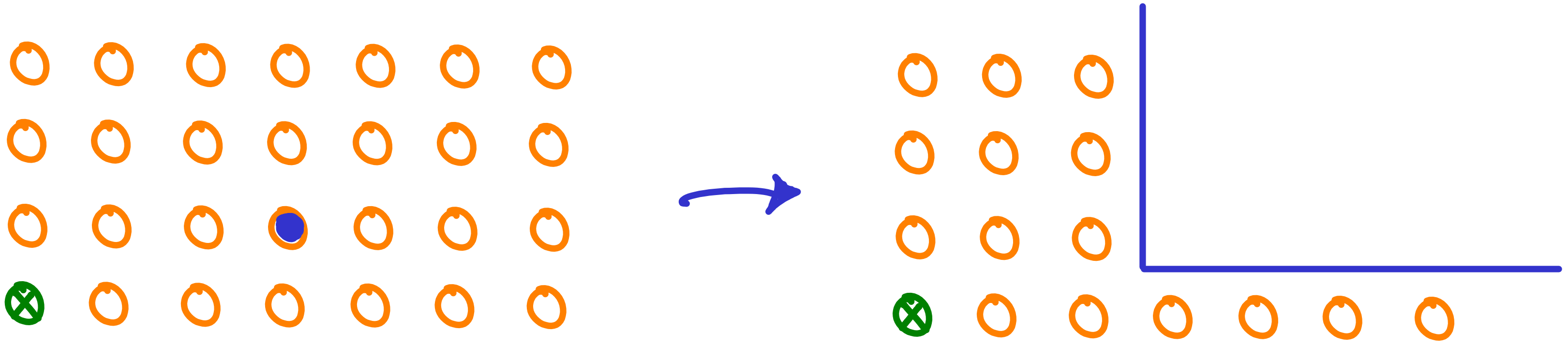

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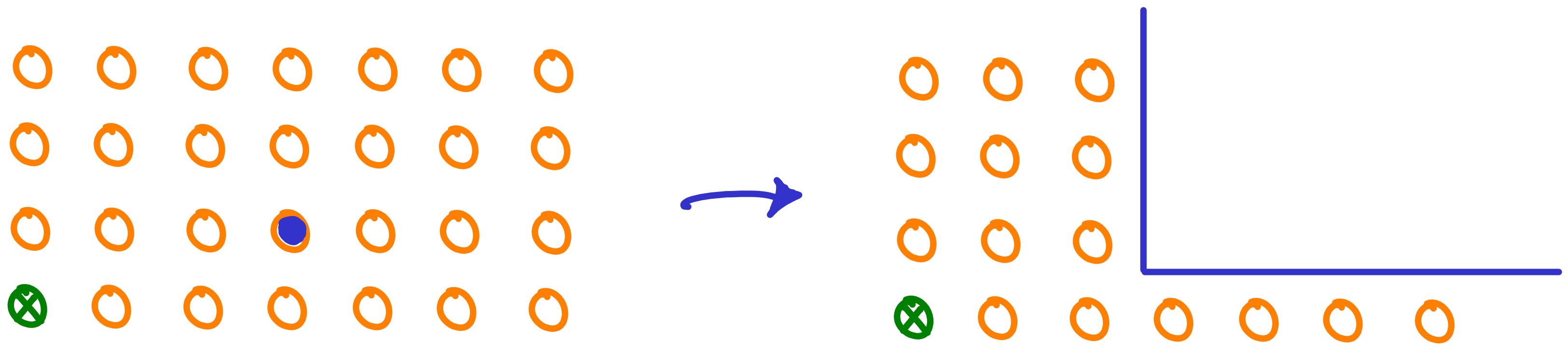

Claim:
there exists a winning strategy for player 1.

Suppose player 1 eats only the top-right cookie.

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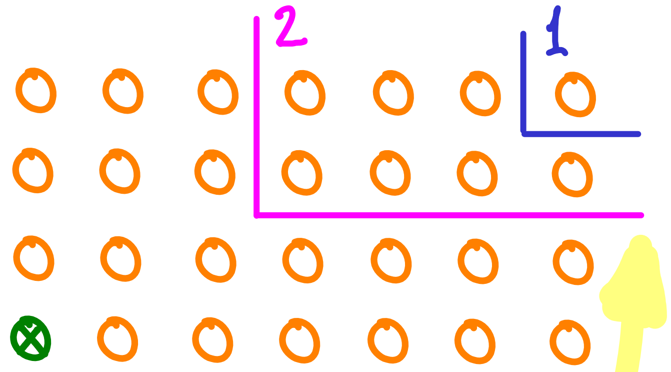

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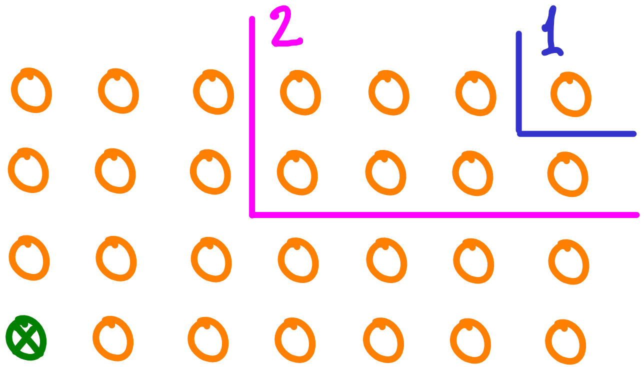

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B) there exists a winning strategy for player 2.

↳ Then player 2 has some first move that ensures a win.

But then player 1 could have made that move and won. □

This proof offers no strategy. No general solution is known.

Other non-constructive proofs:

- A few pigeonhole proofs

  e.g., see proof about disjoint subsets with same sum.

- Every integer $n \geq 2$ is a product of ≥ 2 primes.

  See notes on Induction