Natural numbers : $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ → Some sources: no zero.

Integers : $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Positive integers : $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

Rational numbers : $\mathbb{Q}$ e.g., $\frac{1}{3}, \frac{5}{7}, \frac{13}{2}, 8$

Irrational numbers e.g., $\pi, e, \sqrt{2}$

Real numbers : $\mathbb{R}$ all rational & irrational

Odd vs Even

x is even if and only if $x = 2y$, where y is an integer

x is odd if and only if $x = 2y + 1$, where y is an integer

Prime numbers

An integer x is prime if and only if $x>1$ and $x \neq y \cdot z$
for all positive integers y, z smaller than x.

(x is divisible only by x and 1)

$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

Every integer greater than 1 is prime or composite

$4 = 2 \cdot 2 \qquad 6 = 2 \cdot 3 \qquad 8 = 2 \cdot 4 \qquad 9 = 3 \cdot 3 \qquad \ldots$

Product notation: $x_1 \cdot x_2 \cdot x_3 \cdots x_n = \prod_{i=1}^{n} x_i$

Factorial : $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 \quad = n \cdot (n-1)!$

$$= \prod_{i=1}^{n} i$$

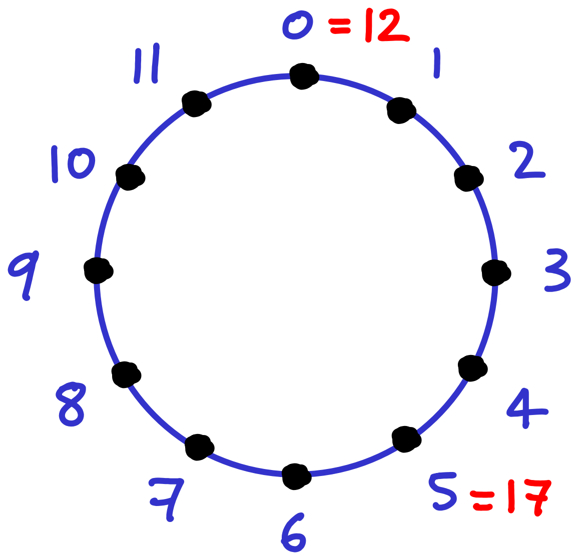
Note: $0! = 1$ by definition

Floor & Ceiling

round down　　　round up

$\lfloor 2.7 \rfloor = 2$　　　$\lceil 2.7 \rceil = 3$

Think of a clock, with 12 positions.



If you start at 0 and travel clockwise...

...5 steps, you get to 5.

...17 steps, you get to 5.

...29 steps, you get to 5.

5, 17, 29, 41, 53, ... are "equivalent".

We write $17 = 5 \pmod{12}$ $53 = 29 \pmod{12}$ etc

$x = y \pmod{n}$ if $|x-y|$ is a multiple of n

Exponent rules

$$b^x = \underbrace{b \cdot b \cdot b \cdots b}_{x \text{ times}}$$

$$b^{1/x} = \sqrt[x]{b}$$

Special case: $b=0$.

$0^x = 0$ if $x>0$

otherwise undefined

$$\boxed{b^{x+1} = b^x \cdot b}$$

$$\boxed{b^{x+y} = b^x \cdot b^y}$$

$$\boxed{b^{x-1} = \frac{b^x}{b}}$$

$$\boxed{b^{x-y} = b^x / b^y}$$

$$\boxed{b^{1-1} = b^0 = 1}$$

$$\boxed{b^{-y} = \frac{1}{b^y}}$$

$$(b \cdot c)^x = b^x \cdot c^x = \underbrace{bc \cdot bc \cdot bc \cdots bc}_{x \text{ times}}$$

$$(b^x)^y = b^{x \cdot y} = \underbrace{b^x \cdot b^x \cdot b^x \cdots b^x}_{y \text{ times}} = \underbrace{(\underbrace{b \cdot b \cdot b \ldots b}_{x \text{ times}}) \cdot (\underbrace{b \cdot b \cdot b \ldots b}_{x \text{ times}}) \cdots (\underbrace{b \cdot b \cdot b \ldots b}_{x \text{ times}})}_{y \text{ times}} \neq b^{(x^y)}$$

Logarithm rules

What does $\log_b x$ mean?

To what power must I raise b, to get x?

The answer is some value y. $x = b^y$

We define $\log_b x = y$

$$\boxed{x = b^y \longleftrightarrow \log_b x = y}$$

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b(b^y) = \log_b x = y$ $\rightarrow$ must raise b to the y to get (b^y)
- $\log_b x^p = \log_b(x^p) = \log_b((b^y)^p) = \log_b(b^{yp}) = yp = p \cdot \log_b x$
- $b^{(\log_b x)} = b^y = x$
- $\boxed{\log_b^p x = (\log_b x)^p \neq \log_b(x^p)}$

- $n^{\log_b a} = a^{\log_b n} \longleftrightarrow \log_b n^{\log_b a} = \log_b a^{\log_b n}$

$$\log_b a \, \log_b n = \log_b n \, \log_b a$$

$$\boxed{x = b^y \quad \longleftrightarrow \quad \log_b x = y}$$

$$\log_b wz = \underbrace{\log_b w}_{s} + \underbrace{\log_b z}_{t}$$

$$\underbrace{s}_{w = b^s} \qquad \underbrace{t}_{z = b^t}$$

$$\log_b wz = \log_b(b^s \cdot b^t) = \log_b b^{s+t} = s + t = \log_b w + \log_b z$$

Similarly, $\log_b w/z = \log_b w - \log_b z$

$$\log_b wz = \log_b w + \log_b z$$

Generalization:

$$\log_b(x_1 \cdot x_2 \cdot x_3 \cdots) = \log_b x_1 + \log_b x_2 + \log_b x_3 + \cdots$$

$$\boxed{\log_b \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} \log_b x_i}$$

Note: $\log_b (wz)^p = p \log_b wz = p \log_b w + p \log_b z = \log_b w^p + \log_b z^p$

but $\log_b^p (wz) \neq \log_b^p w + \log_b^p z$

$$\boxed{x = b^y \longleftrightarrow \log_b x = y}$$

$$\log_r x = \underbrace{\log_r b^y}_{} = y \log_r b \Rightarrow y = \frac{\log_r x}{\log_r b}$$

$$\boxed{\log_b x = \frac{\log_r x}{\log_r b}}$$

(for any r)

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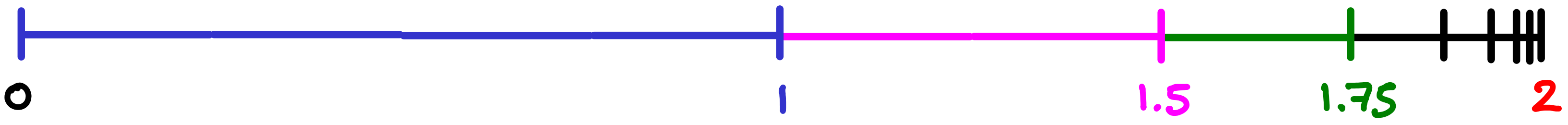
choose $r = x$...

$$\log_b x = \frac{\log_x x}{\log_x b} = \frac{1}{\log_x b}$$

If you take a unit step, then half of that, then half of the half etc, always taking half the previous step,

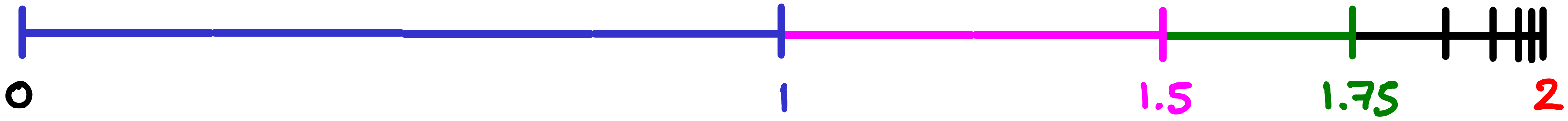
you will not travel more than 2 units.

(you only reach "2" in the sense of a limit)



$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots+\frac{1}{2^i}\dots=\sum_{i=0}^{\infty}\frac{1}{2^i}\le 2$$ (equal to 2 as limit)

Zeno: if you always take steps equal to half the remaining distance, you will never reach the target.



The length of the first step is at least the sum of all other steps.

If the ratio between steps is larger than 2 $\left(\text{e.g., } \underbrace{1, \frac{1}{4}, \frac{1}{16}, \dots, \frac{1}{4^i}}_{\text{ratio} = 4}\right)$

the statement above still holds.

If the ratio is smaller but equal to a constant > 1,

$\text{ratio} = \frac{10}{9} \rightarrow \left(\text{e.g., } 1, \frac{9}{10}, \frac{81}{100}, \dots, \frac{1}{(10/9)^i}\right)$

then all steps sum to at most some constant times the first. ...

Geometric series

$$\sum_{i=0}^{\infty} \frac{1}{\alpha^i}$$ for some fixed value $\alpha > 1$

- for $\alpha = 2$, $\sum = 2$
- in general, $\sum = \dfrac{1}{1 - 1/\alpha}$

Most sources:

$r = \dfrac{1}{\alpha}$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \quad \text{for} \quad 0 < r < 1$$

Now this is the "first step"

$$\sum_{i=0}^{n} 2^i = \underbrace{1+2+4+8+16+\cdots}_{<2^n} + 2^n < 2 \cdot 2^n$$

Every term is a fixed percentage of the next. (e.g. 50%)

↳ All terms sum to some constant times the largest.

This is used <u>all the time</u> in algorithmic analysis

Harmonic numbers

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

grows roughly like $\log_e n$

Arithmetic series

$$\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$$

proved several ways in course notes

Telescoping series

If you have n+1 numbers, F_i $(0 \le i \le n)$

or a function F in general, then

$$\sum_{i=1}^{n} (F_i - F_{i-1}) = F_n - F_0$$

$$\begin{aligned}
i=n: \quad & F_n - \cancel{F_{n-1}} \\
& + \cancel{F_{n-1}} - \cancel{F_{n-2}} \\
& + \cancel{F_{n-2}} - F_{n-3} \\
& \quad \vdots \\
& + F_2 - \cancel{F_1} \\
i=1: \quad & + \cancel{F_1} - F_0
\end{aligned}$$