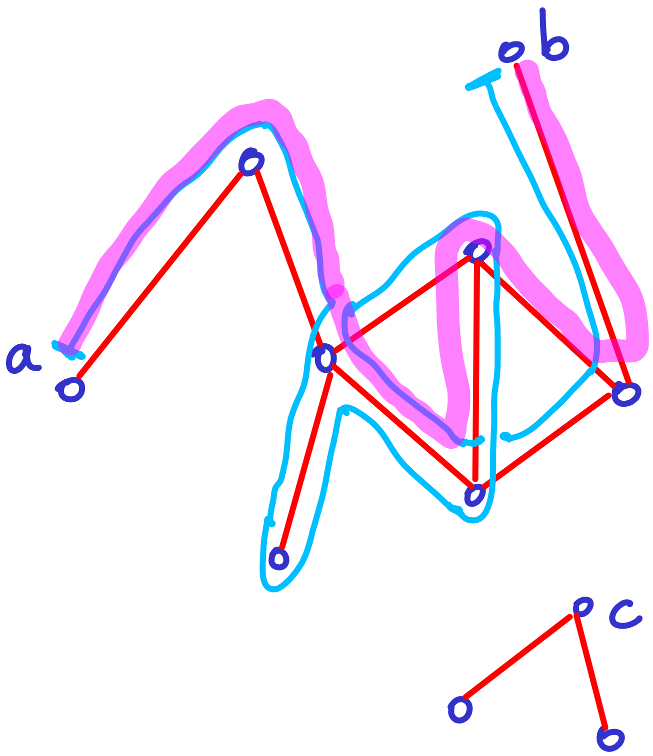
# GRAPH CONNECTIVITY




A walk is a sequence of vertices $v_i, v_{i+1}, v_{i+2}, \ldots, v_k$

s.t. every $v_j, v_{j+1}$ is an edge

$(i \leq j < k)$

We can walk from a to b, but not from a to c.

A path is a walk with distinct vertices

# GRAPH CONNECTIVITY

A vertex x is connected to a vertex y (in G)
if there is a path from x to y

could say "walk"

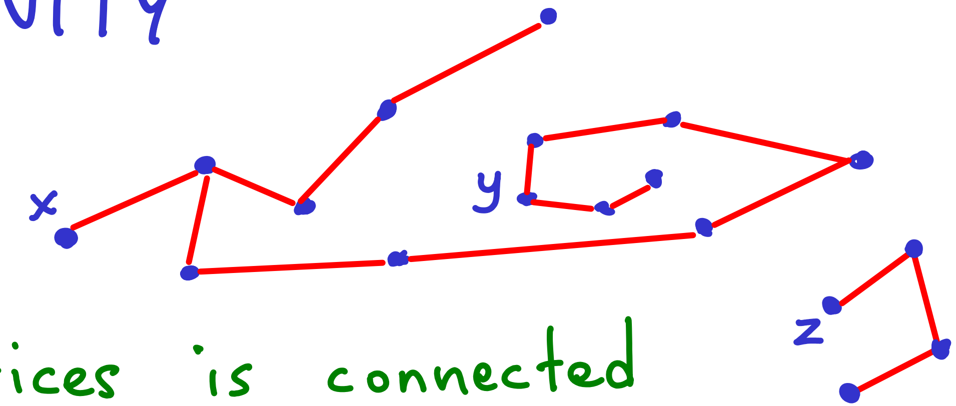
If there is an x-y walk in G then there is an x-y path in G.

↳ Pick the shortest x-y walk that is not a path.

It must have a repeated vertex, u} $x \ldots u \underbrace{\ldots}_{\text{remove}} u \ldots y$

CONTRADICTION

# GRAPH CONNECTIVITY



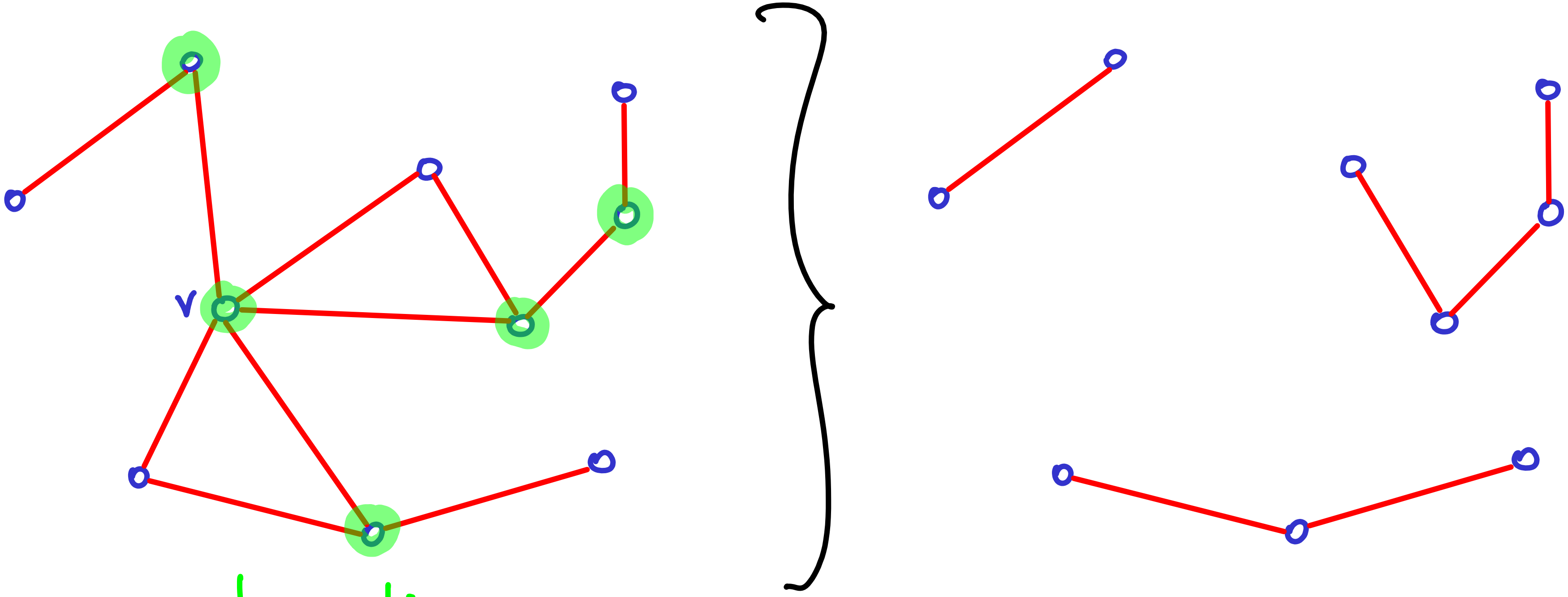

A graph is connected if every pair of vertices is connected

If vertex x is connected to vertex y then they are in the same component.

If x & y are in the same component
but x & z are not
then y & z are not.

Given $G$, remove a vertex: $G-v$

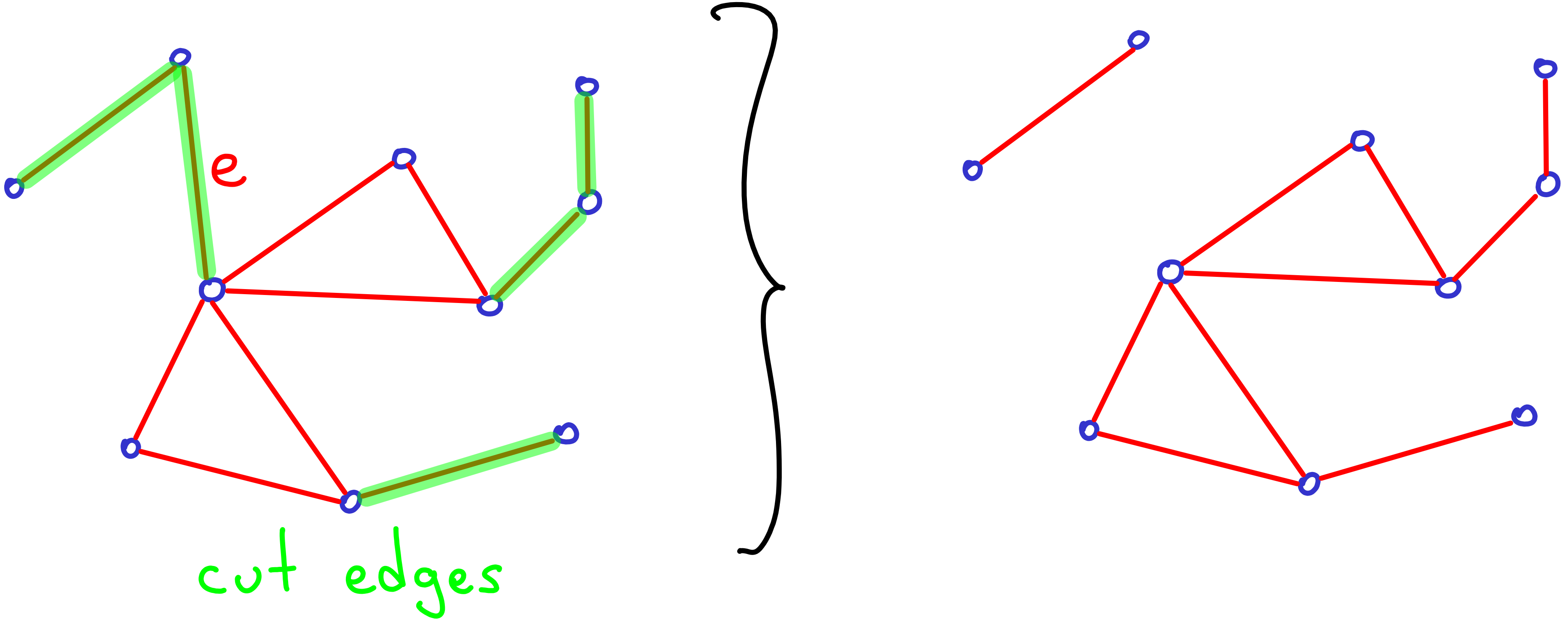
If $G-v$ has more components than $G$, then $v$ is a cut vertex.




cut vertices

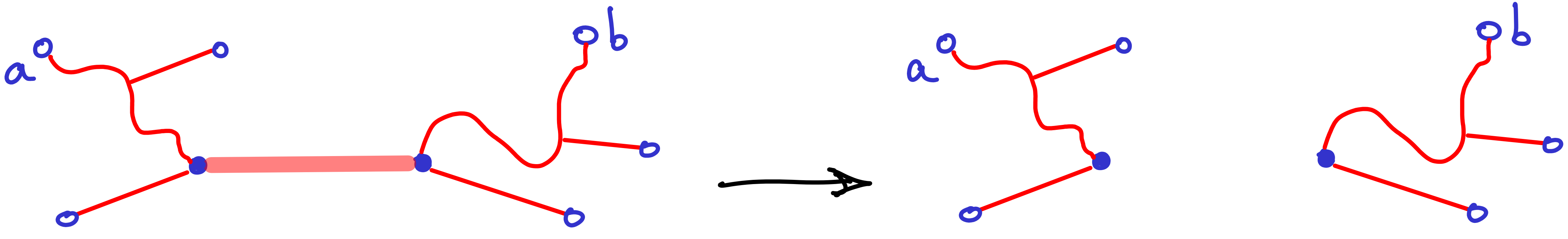
Given G, remove an edge : G - e

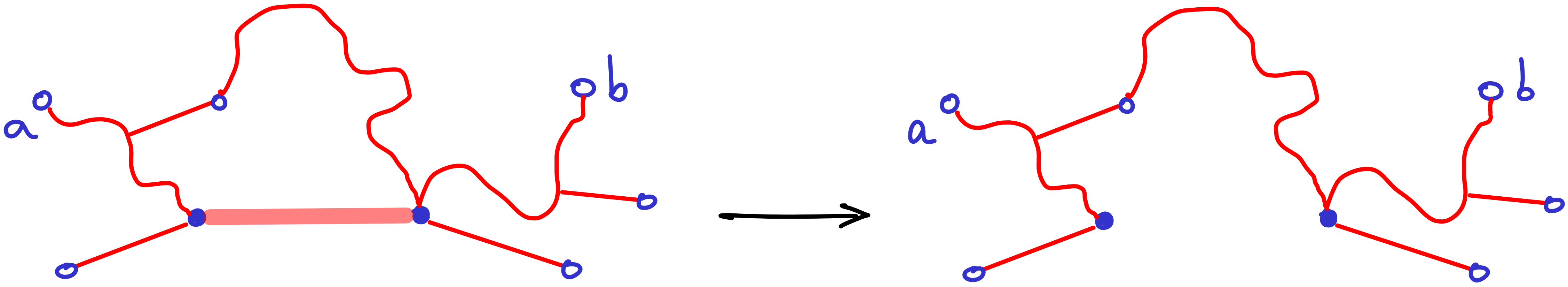
If G-e has more components than G, then e is a cut edge.



Claim: a cut edge can't be on a cycle.

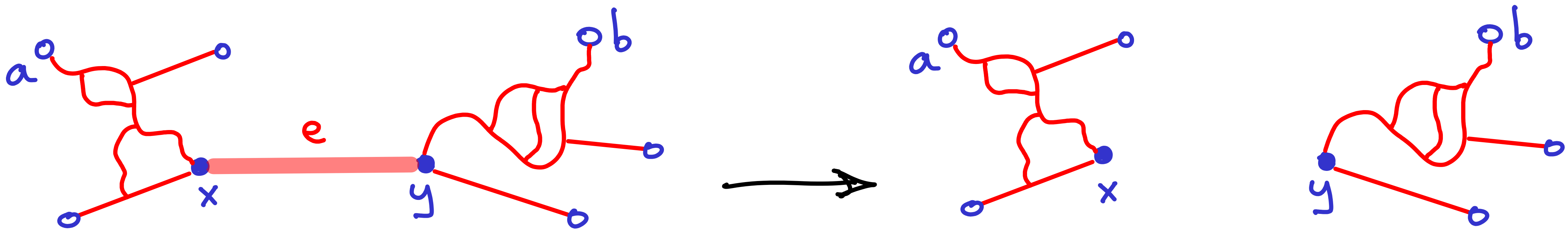
(a cycle is a path w/ start = end)



...cont'd: if $e$ is a cut edge then $\exists\ a, b$ s.t. $a$ is not connected to $b$ in $G-e$.

So all paths from $a$ to $b$ in $G$ use $e$, i.e traverse $a \cdots xy \cdots b$ where $e = \overline{xy}$.



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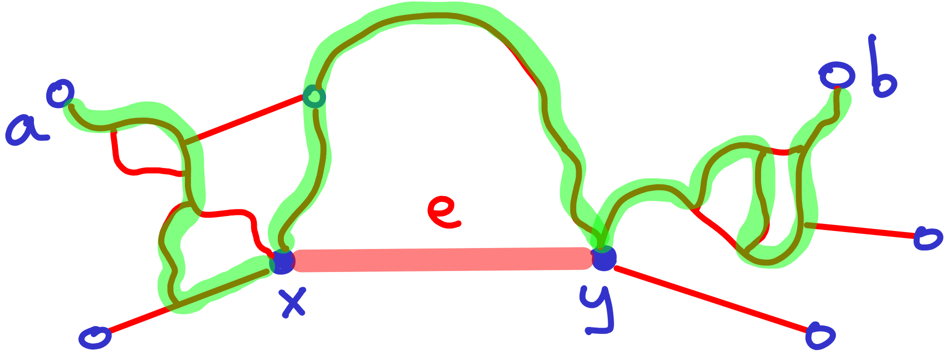



But if $e$ is on a cycle then we can walk from $a$ to $x$, walk from $x$ to $y$, without using $e$, and walk from $y$ to $b$.

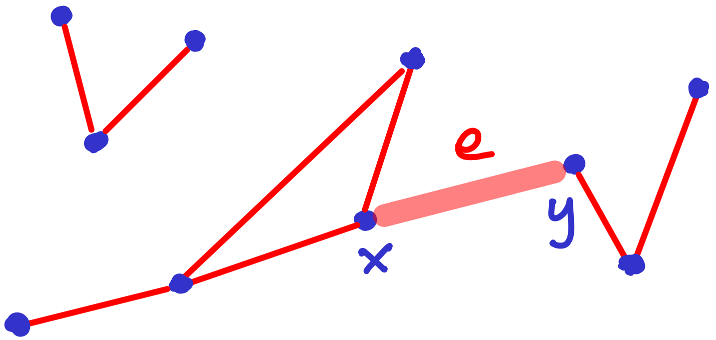
CONTRADICTION

Claim: Removing a cut edge $e=(x,y)$ increases the number of components by 1.

Types of vertex pairs $(a,b)$ in $G$:

0) no path exists between $a$ & $b$
1) not all paths between $a$ & $b$ use $e$
2) all paths use $e$

(path exists)



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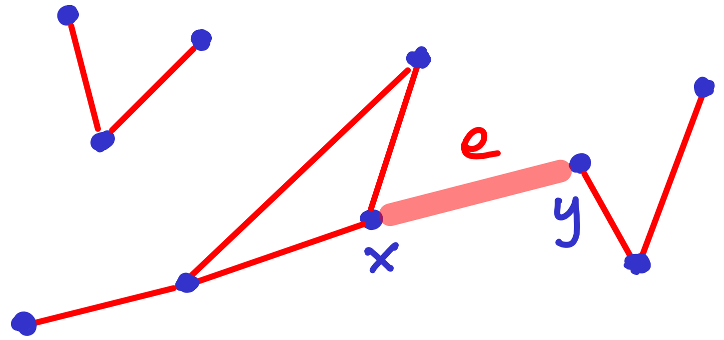
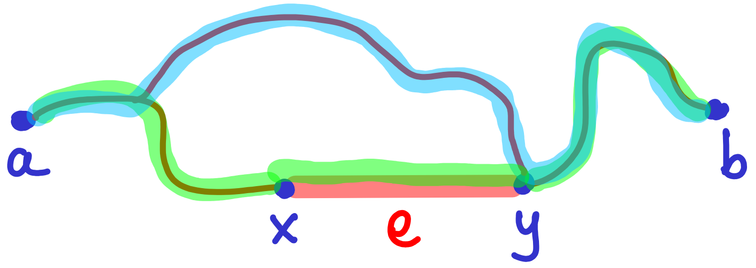



CONTRADICTION

If path P uses e
& path Q doesn't
then e is on a cycle

but cut edges don't exist on cycles



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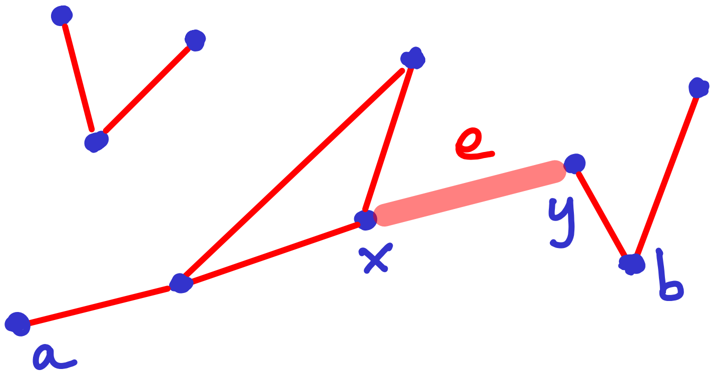


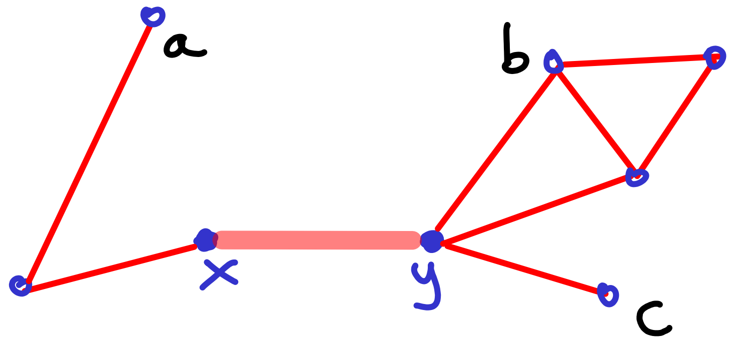

Type 0 or 1: not affected by removal of $e$

Type 2: implies $a$ connects to $x$ but not to $y$ & $b$ connects to $y$ but not to $x$ $\Big\}$ in $G-e$

↳ Type 2 partitions one component into two.

**Claim:** Removing a cut edge $e = (x, y)$ increases the number of components by 1.

* $e$ can only affect the component it's in.
So focus on connected graphs.




Proof by contradiction.

Suppose $G - e$ has $\geq 3$ components. $\exists\, a, b, c$ in different components.

In $G$, all paths $a \to b$ use $e$ } wlog $a \to x \to y \to b$

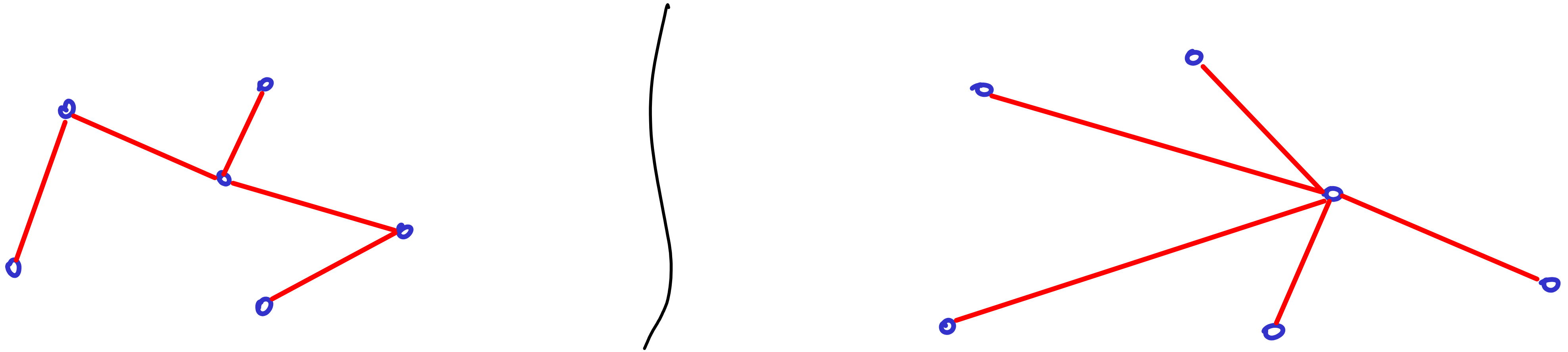
all paths $a \to c$ use $e$ { if $a \to x \to y \to c$

$\Rightarrow$ $b$ & $c$ in same component

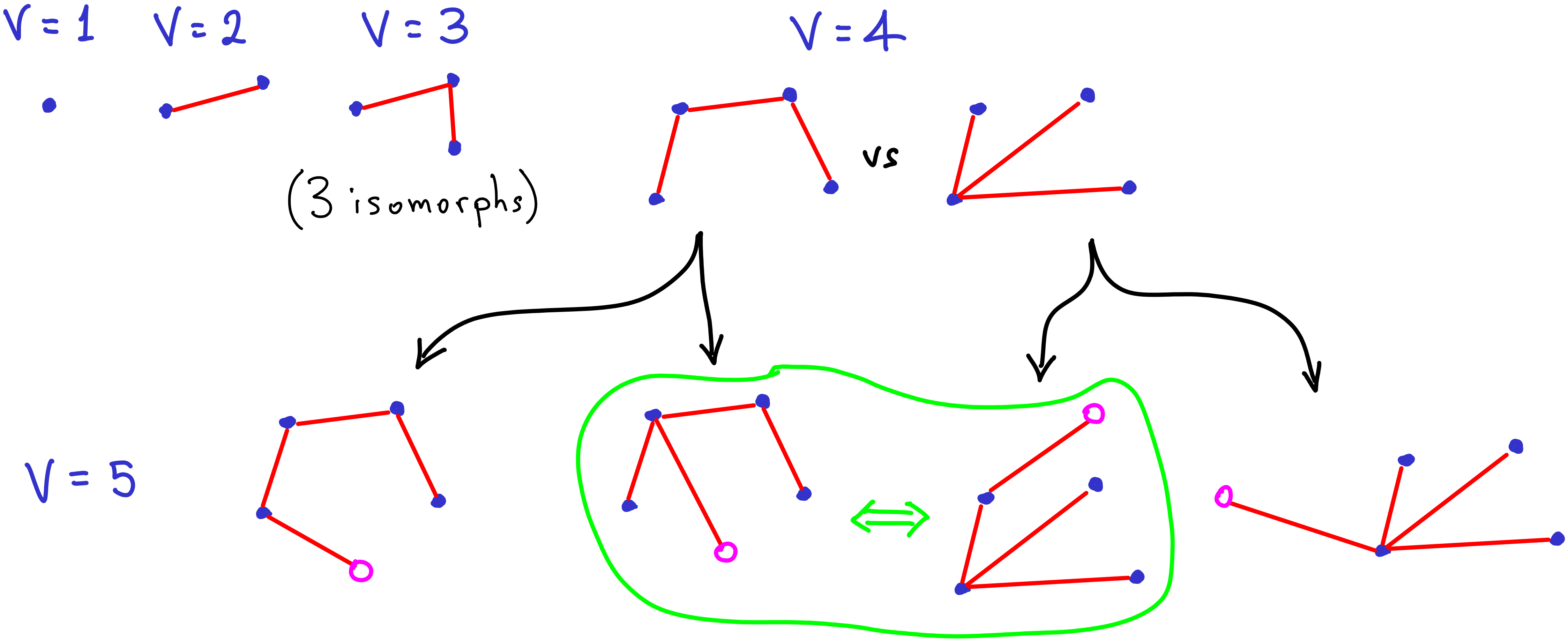
if $a \to y \to x \to c$

$\Rightarrow$ $e$ not a cut edge

TREES : CONNECTED ACYCLIC GRAPHS



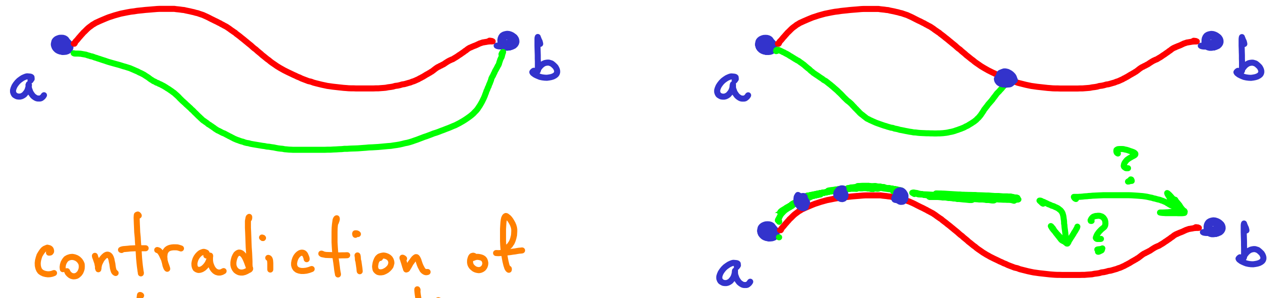
FORESTS : ACYCLIC GRAPHS (collections of trees)


V=1
V=2
V=3
(3 isomorphs)
V=4
vs
V=5

tree $\iff$ there is a unique path between every pair of vertices

$\Rightarrow$ • for any vertices $a, b$ : a path exists (trees are connected)

• suppose $\geq 2$ paths.




cycle : contradiction of tree: acyclic

$\Leftarrow$ • if for every 2 vertices a path exists, then graph is connected

• if any 2 vertices are on a cycle, then they are on $\geq 2$ paths but we assume unique paths, so no 2 vertices are on a cycle.

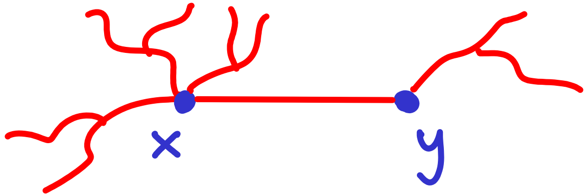
↳ acyclic

$\square$

For any connected graph,

tree $\iff$ every edge is a cut edge

$\Rightarrow$




for any $x, y$

tree $\Rightarrow$ unique path from $x$ to $y$

$\Rightarrow$ $\overline{xy}$ : cut edge

$\Leftarrow$ suppose graph $\neq$ tree.

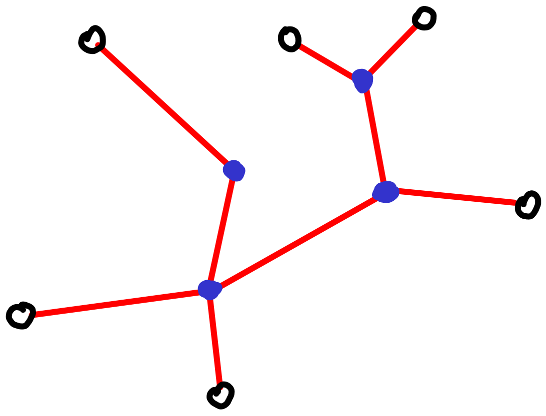
Then it has a cycle.

We have proved: A cut edge can't be on a cycle.

not every edge is a cut edge. (CONTRADICTION)

# LEAVES : vertices of degree 1

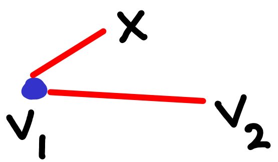
If $V \geq 2$, then T has $\geq 2$ leaves



Consider longest path in T. $v_1 \dots v_k$

($k \geq 2$)

If $v_1 \neq$ leaf, then




$x \neq v_i$ (not on path)

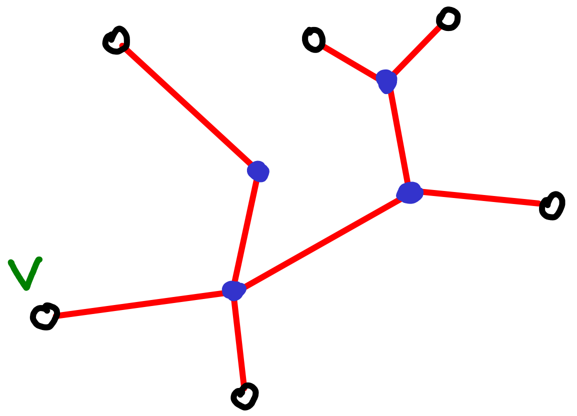
($x = v_i$ would create cycle)

Then $x v_1 \dots v_k$ : longer path : CONTRADICTION

So $v_1$ & $v_k$ : leaves

# If v is a leaf in tree T, then T−v is a tree

- removing v doesn't create cycles.
- removing v doesn't disconnect.
  ($v \neq$ cut vertex ; T−v is connected)




↳ if v were a cut vertex, then $\exists\, a, b$ ($a \neq v$, $b \neq v$) s.t.
any path $a \to b$ must use v.
But v is a dead end:
can't be part of such a path

in fact, "the only path" (T: unique paths)

## If $v$ is a leaf in tree $T$, then $T-v$ is a tree

This allows us to use induction

ex: if $|V(T)| = n \geqslant 2$ then $|E(T)| = n-1$

pf: Base case: $n=2$  trivial

Hypothesis: for $2 \leq k < n$, statement holds.

Suppose $T$ has $n$ vertices. Find a leaf $v$ & delete.

- $v$ had degree 1, so we delete 1 edge.
- $T-v$ is a tree, w/ $n-1$ vertices $\to$ $n-2$ edges.
- Replace $v$: total edges $= n-2+1 = n-1$

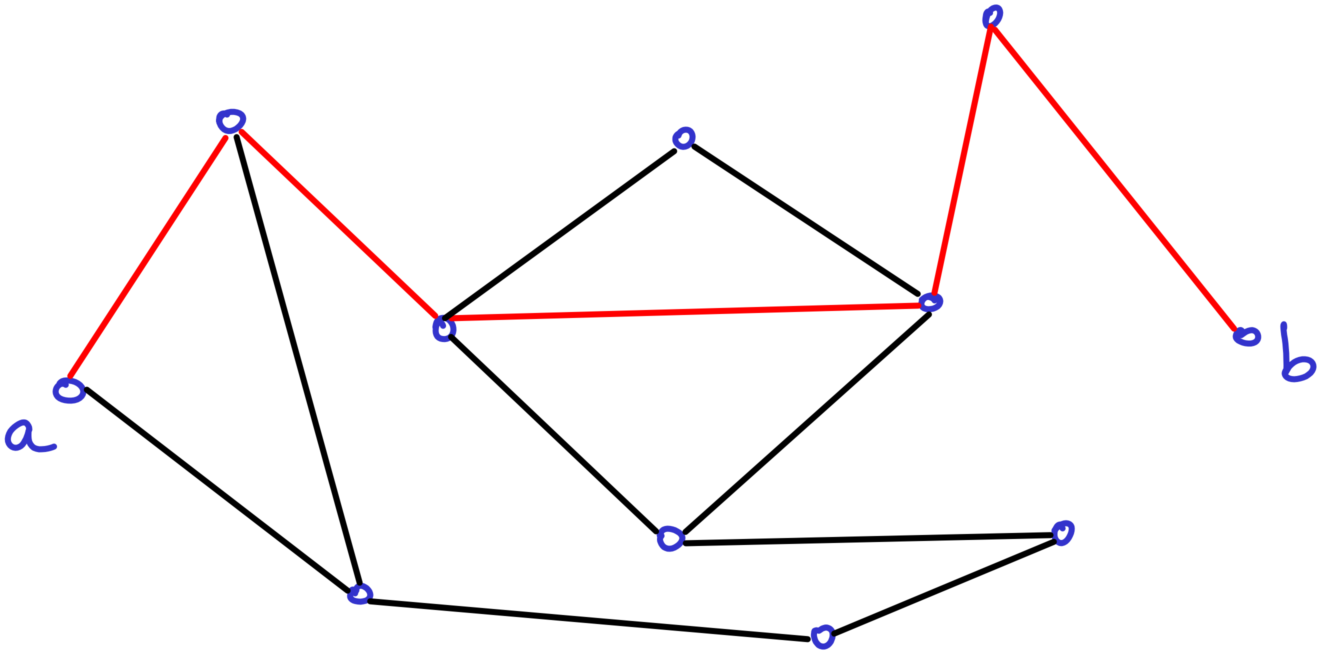
Proved: if $|V(T)| = n \geq 2$ then $|E(T)| = n-1$

Also true: for connected $G$ with $n \geq 1$ vertices,
if $|E(G)| = n-1$ then $G$ is a tree

See p. 354

Also defines spanning trees

# DISTANCE IN GRAPHS




$d(a,b) = 5$

$d(a,b)$ = length of shortest path between a & b

Find minimum distance: covered in Algorithms course