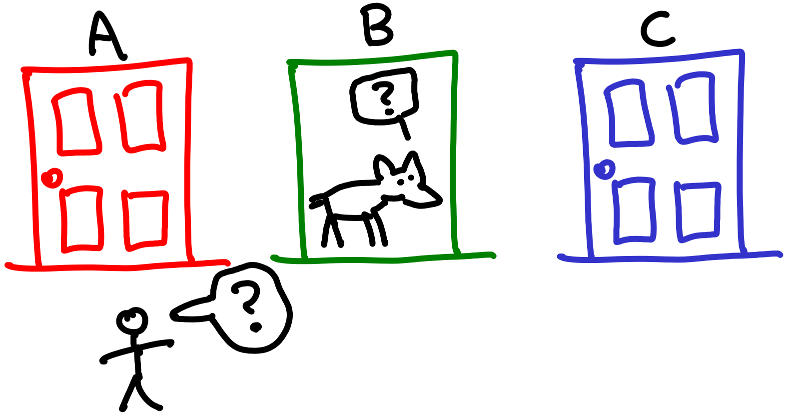
THE MONTY HALL PROBLEM



1 of these 3 doors hides a car.
The other 2 hide goats.

You get to pick a door. You randomly pick A.

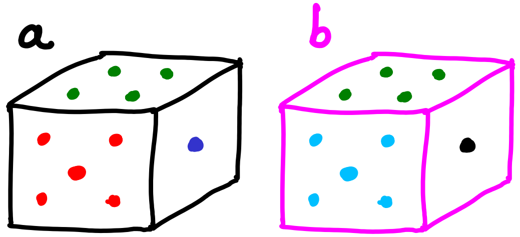
Then a door you didn't pick is opened (say, B) revealing a goat.

You're given the choice: KEEP YOUR DOOR OR SWITCH

?

CONDITIONAL PROBABILITY

Roll 2 dice ...



$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$

"prob. A given B"

$P(A \mid B)$

If we knew that both are even, then what is the probability that the sum is 8?

$A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$B = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6)\}$

$P(A|B) = \frac{3}{9}$

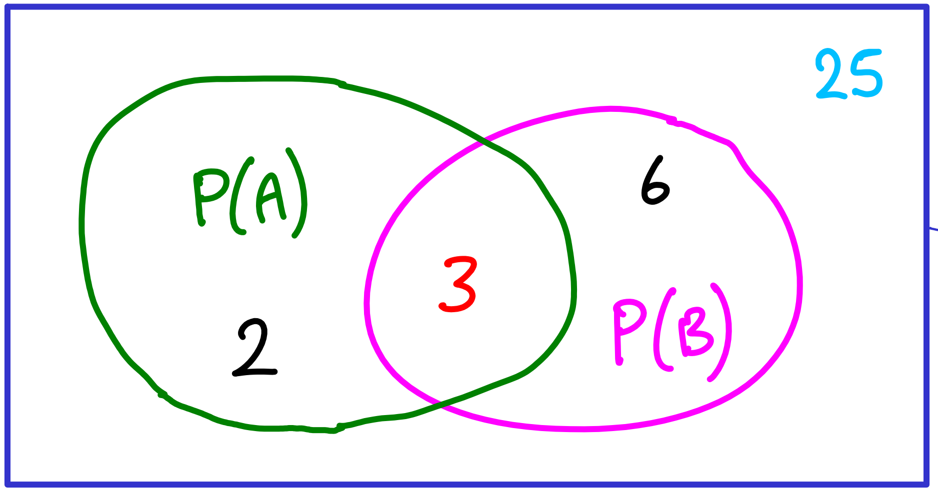
$\neq P(A)$

in this example

$$P(A) = P(\text{sum} = 8)$$

$$P(B) = P(\text{both are even})$$

$$P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36}$$



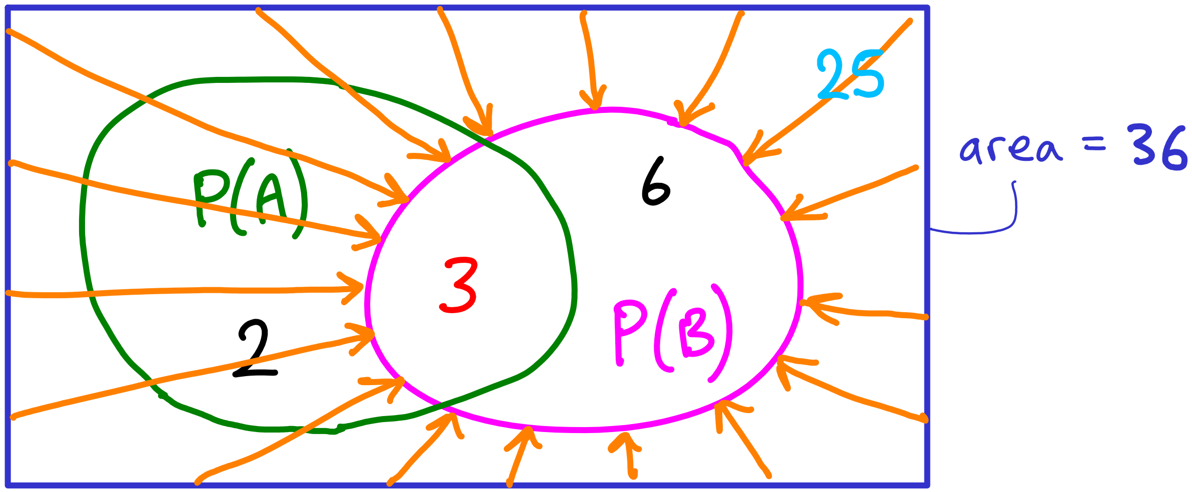
area = 36

Divide all these numbers by 36, if you prefer the universe to have area 1.

$P(A) = P(\text{sum} = 8)$

$P(B) = P(\text{both are even})$

$$P(A) = \frac{\text{green area}}{\text{blue area}} = \frac{5}{36}$$



When we establish B then the universe shrinks.

The probability that A holds is normalized: $\dfrac{\text{remaining valid green area}}{\text{new universe (pink area)}}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3}{3+6}$$

another example

Flip a coin 5 times. $P(\text{1st flip} = T) = \frac{1}{2}$

But what if you know that 3 of the 5 flips were H ?

$$P(\text{1st flip} = T \mid 3\cdot H) = \frac{P[(\text{1st flip} = T) \cap (3\cdot H)]}{P(3\cdot H)}$$

$$P(3\cdot H) = \frac{\binom{5}{3}}{2^5} \quad \begin{matrix} \to \text{Ways to choose 3 positions for H.} \\ \to \text{Sample space.} \end{matrix} \quad = \frac{\frac{5!}{3!2!}}{32} = \frac{5}{16}$$

$$\to \frac{2/16}{5/16} = \boxed{\frac{2}{5}}$$

$$P[(\text{1st flip} = T) \cap (3\cdot H)] : \left.\begin{matrix} T\ HHHT \\ T\ HHTH \\ T\ HTHH \\ T\ THHH \end{matrix}\right\} \frac{4}{32} \quad \text{OR} \quad \frac{1}{2} \cdot \frac{\binom{4}{3}}{2^4} = \frac{1}{8}$$

another example

Flip a coin 5 times. $P(\text{1st flip} = T) = \frac{1}{2}$

But what if you know that 3 of the 5 flips were H ?

think of having a biased coin : 60-40 vs 50-50

then the first flip has 40% for T $\rightarrow \frac{2}{5}$

P(≥ 2 people in a group of k have same birthday)

= 1 - P(all k have distinct birthdays)

↳ P(2nd person has different bday than 1st) $= \frac{364}{365}$ = P(A)

• $\underbrace{\text{P(3rd . . . . 1st \& 2nd)}}_{B}$ → $\underbrace{\frac{363}{365} \ \boxed{= P(B|A)}}_{\text{assuming 1st \& 2nd differ}}$

• P(4th . . . . (1-3)) → $\frac{362}{365}$ = P(C | (A∩B))

etc

= P(A) ∩ P(B|A) ∩ P(C|(A∩B)) ...

INDEPENDENCE

Flip a coin x3 : $P(3rd = T \mid 1st = H) =$

$$= \frac{P[(3rd = T) \cap (1st = H)]}{P(1st = H)} = \frac{\frac{2}{8}}{\frac{1}{2}} = \frac{1}{2}$$

2 → # outcomes

8 → sample space

Notice $P(3rd = T) = \frac{1}{2}$ so knowledge of $(1st = H)$ was useless.

A & B are independent if $P(A) = P(A \mid B)$

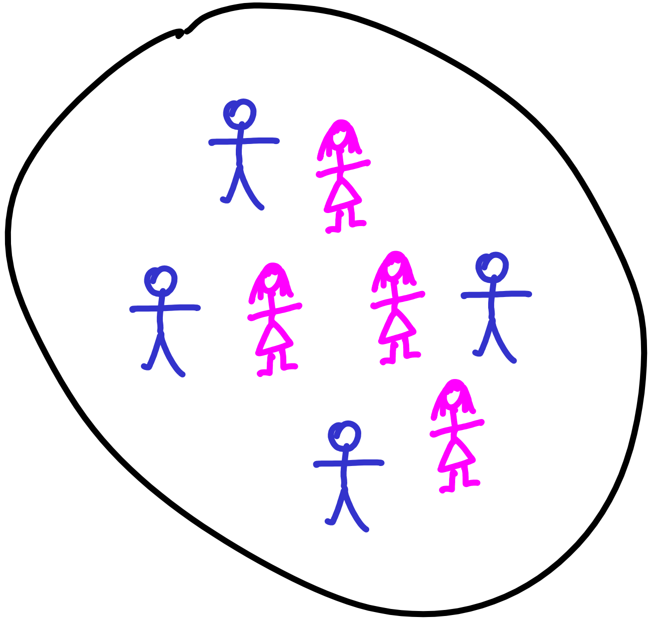
if $P(B) = P(B \mid A)$ [equivalent]

$$\underbrace{\underbrace{P(A|B) = \frac{P(A \cap B)}{P(B)}}_{\text{always}} = P(A)}_{\text{if } A \,\&\, B \text{ independent}}$$

alternate definition: A & B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

another example



4 boys & 4 girls

Select 2 people from this group

A: 1st person is a girl

B: 2nd person is a girl

dependent

$$P(A) = \frac{4}{8}$$

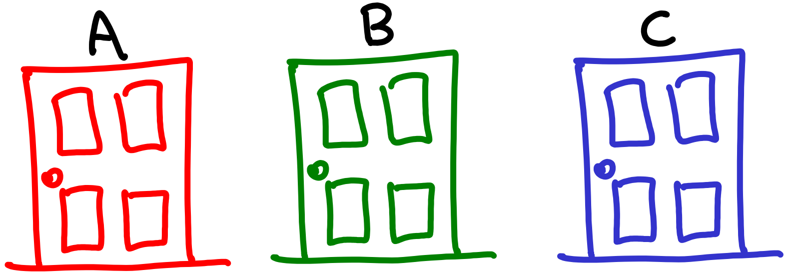
$$P(B) = \frac{1}{2}$$ (by symmetry)

(or: sample space = $8 \cdot 7$
& for each girl = 2nd, #outcomes = 7)

$\neq$

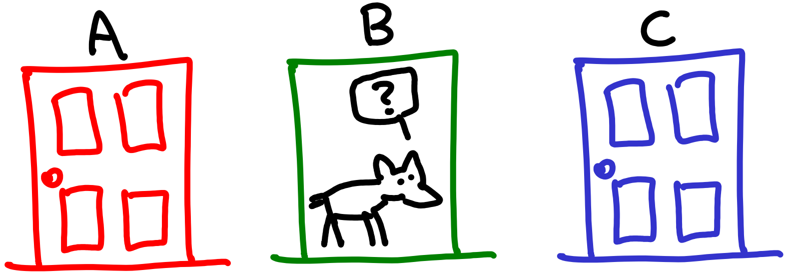
$$P(B|A) = \frac{3}{7}$$

BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

Now we know the car is not at B
What is the probability it's at A?

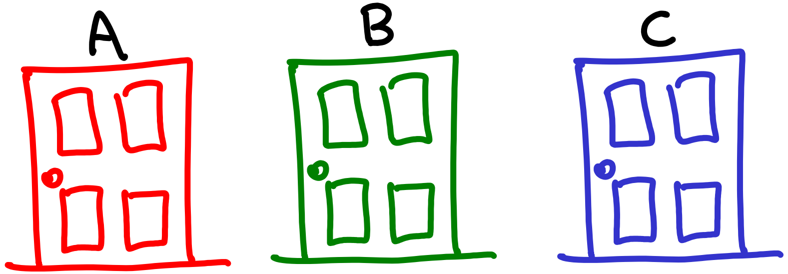


$$P(A \mid \bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$

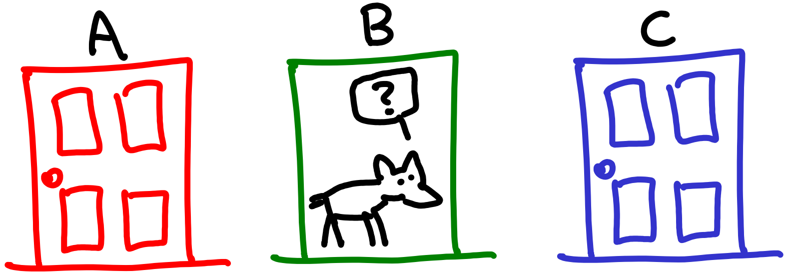
...because $P(\bar{B} \mid A) = \dfrac{P(A \cap \bar{B})}{P(A)}$ ...

$$= \frac{P(\bar{B} \mid A) \cdot P(A)}{P(\bar{B})} = \frac{1 \cdot P(A)}{P(\bar{B})} = \frac{1/3}{2/3} = \frac{1}{2}$$

BACK TO MONTY HALL



$$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$$

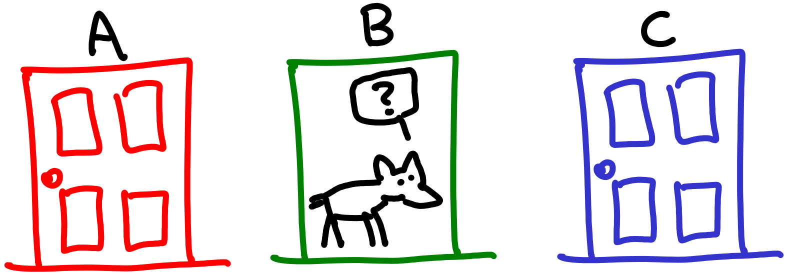


$\cancel{P(A \mid \overline{B})}$ ← back up

What we actually want is

$$P(A \mid (\underbrace{\text{door B was opened}}_{\text{not} = \overline{B}} \cap \underbrace{\text{we chose A}}_{\text{extra info}}))$$

BACK TO MONTY HALL : intuition



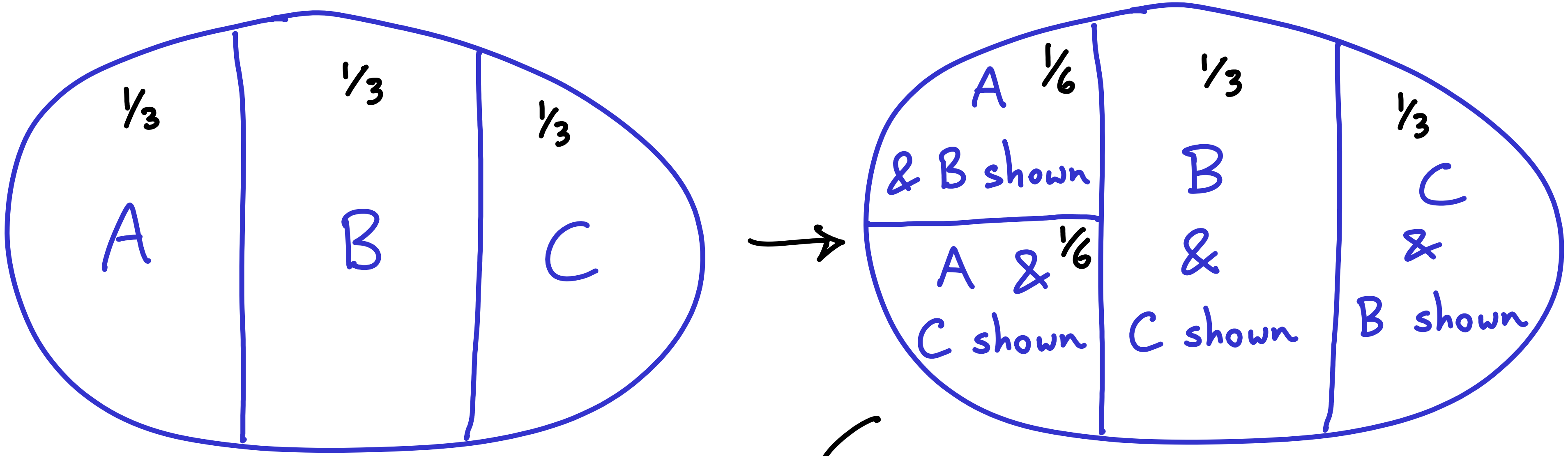
$P(\text{car @ } A) = P(A) = \frac{1}{3} = P(B) = P(C)$

w.l.o.g. guess A
& B is shown.

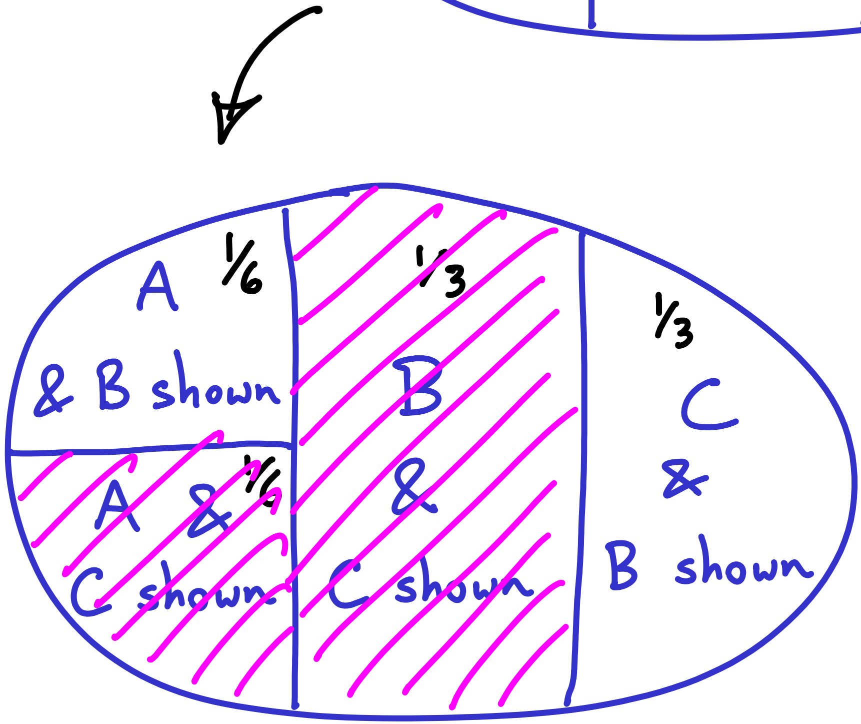
$\underbrace{\text{4 events}}$

- $\xrightarrow{1/3}$ if A (car @ A) then
  - $\xrightarrow{1/2}$ B revealed $\rightarrow \frac{1}{3} \cdot \frac{1}{2}$
  - $\xrightarrow{1/2}$ C revealed

  } switch = lose
- $\xrightarrow{1/3}$ if B, then C revealed → switch to win
- $\xrightarrow{1/3}$ if C, then B revealed → switch to win $\rightarrow \frac{1}{3}$

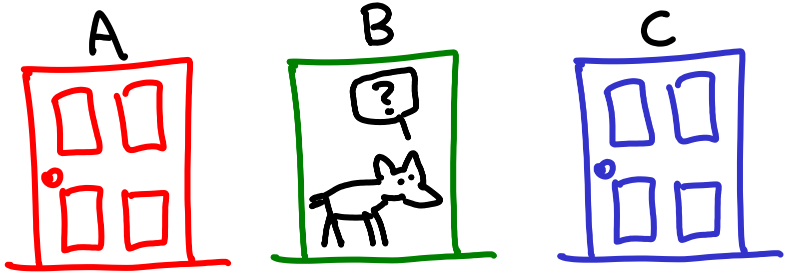


When we establish that B is shown, the universe shrinks.



$$P(A \cap B \text{ shown}) = \; = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

$$P(C \cap B \text{ shown}) = \; = \frac{1/3}{1/6 + 1/3} = \frac{2}{3}$$



$$P\big[A \mid (\text{we chose } A \cap \text{door } B \text{ was opened})\big]$$

$$= \frac{P\big[A \cap (\text{we chose } A \cap \text{door } B \text{ was opened})\big]}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

apply $P(x|y) = \dfrac{P(x \cap y)}{P(y)}$

just moving parentheses

$$= \frac{P\big[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\big]}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

again, for numerator

$$= \frac{P\big[\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)\big] \cdot P(A \cap \text{we chose } A)}{P(\text{we chose } A \cap \text{door } B \text{ was opened})}$$

So far,

$$P\left[A \mid (\text{we chose A} \cap \text{door B was opened})\right]$$

$$= \frac{P\left[\text{door B was opened} \mid (A \cap \text{we chose A})\right] \cdot P(A \cap \text{we chose A})}{P(\text{we chose A} \cap \text{door B was opened})}$$

$$= \frac{\overbrace{1/2}^{\text{host picks randomly}} \cdot \overbrace{P(A) \cdot P(\text{we chose A})}^{\text{independent}}}{P(\text{we chose A} \cap \text{door B was opened})}$$

$$= \frac{1/2 \cdot (1/3 \cdot 1/3)}{P(\text{we chose A} \cap \text{door B was opened})}$$

So far,

$$P\left[A \mid (\text{we chose A} \cap \text{door B was opened})\right]$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3}}{P(\text{we chose A} \cap \text{door B was opened})}$$

The 2 terms in the denominator are "mutually exclusive", which means that they have no intersection. We can see that because one asks for A to happen, but the other asks for C to happen (among other things). In fact we should also include the term +P(we chose A AND door B was opened AND B) but this term is equal to zero.
See full notes for a little more discussion on this.

$$= \frac{1/18}{\left[\begin{array}{l} P(\text{we chose A} \cap \text{door B was opened} \cap A) \\ +P(\text{we chose A} \cap \text{door B was opened} \cap C) \end{array}\right]}$$

could not have door B opened AND car at B

$$P(\text{open B} \mid (C \cap \text{choose A})) = 1 = \frac{P(C \cap \text{open B} \cap \text{choose A})}{P(C \cap \text{choose A})}$$

$$= \frac{1/18}{P(\text{we chose A} \cap \text{door B was opened} \cap A) + \frac{1}{3}\cdot\frac{1}{3}}$$

So far,

$$P\left[A \mid (\text{we chose } A \cap \text{door } B \text{ was opened})\right]$$

$$= \frac{1/18}{P(\text{we chose } A \cap \text{door } B \text{ was opened} \cap A) + \frac{1}{3} \cdot \frac{1}{3}}$$

$$= \frac{1/18}{P\left[(A \cap \text{we chose } A) \cap \text{door } B \text{ was opened}\right] + 1/9}$$

$$= \frac{1/18}{P\left[\underbrace{\text{door } B \text{ was opened} \mid (A \cap \text{we chose } A)}_{\text{host's random choice}}\right] \cdot \underbrace{P(A \cap \text{we chose } A)}_{\text{independent}} + 1/9}$$

$$= \frac{1/18}{1/2 \cdot (1/3 \cdot 1/3) + \frac{1}{9}} = \frac{1/18}{1/18 + 2/18} = \boxed{\frac{1}{3}}$$

TESTING FOR A DISEASE

- Suppose 1% of the population has a disease
- there is a diagnostic test, that finds it 80% of the time (assuming the subject has it)
- the test also produces false positives, at a rate of 9.6% (you're fine, but the test says you're not)

If someone tests positive,
what are the odds that they have the disease?

4 events

	1% Have disease	99% Don't have disease
Test ☹	80%	9.6%
Test ☺	20%	90.4%

$$P(\text{disease} \mid \text{test} ☹) = \frac{P(\text{disease} \cap \text{test} ☹)}{P(\text{test} ☹)} = \frac{P(\text{test} ☹ \mid \text{disease}) \cdot P(\text{disease})}{P(\text{test} ☹)}$$

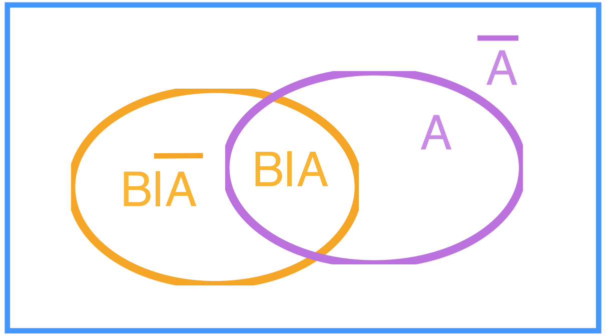
$$P(\text{test} ☹) = \begin{cases} P(\text{test} ☹ \mid \text{disease}) \cdot P(\text{disease}) = 0.8 \cdot 0.01 = 0.008 \ (0.8\%) \\ + \\ P(\text{test} ☹ \mid \text{no disease}) \cdot P(\text{no disease}) = 0.096 \cdot 0.99 \simeq 0.095 \ (9.5\%) \end{cases}$$

$$P(\text{disease} \mid \text{test} ☹) = \frac{0.008}{0.008 + 0.095} \sim 7.8\%$$

Bayes theorem

$$P(A \cap B) = P(A|B) \cdot P(B)$$
$$= P(B|A) \cdot P(A)$$

$$P(A|B) = P(B|A) \cdot \frac{P(A)}{P(B)}$$



$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})}$$

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B|A)\cdot P(A) + P(B|\bar{A})\cdot P(\bar{A})}$$

A: have disease

B: test

1% A	99% $\bar{A}$
$B|A$ = 80%	$B|\bar{A}$ = 9.6%

$P(A) = 0.01$

$P(\bar{A}) = 0.99$

$P(B|A) = 0.8$

$P(B|\bar{A}) = 0.096$

$$P(A|B) = \frac{0.8\cdot 0.01}{0.8\cdot 0.01 + 0.096\cdot 0.99}$$